YOUR PRACTICE PAPER

ANALYSIS AND APPROACHES

HIGHER LEVEL
FOR IBDP MATHEMATICS

ANSWERS

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- 4 Sets of Practice Papers
- Distributions of Exam Topics
- Exam Format Analysis
- Comprehensive Formula List

SE PRODUCTION LIMITED

AA HL Practice Set 1 Paper 1 Solution

Section A

1. (a) The mean

$= \frac{300}{15}$ (M1) for valid approach

$= 20$ A1

[2]

(b) (i) -40 A1

(ii) The new variance

$= (-2)^2(9)$ (M1) for valid approach

$= 36$ A1

(iii) 6 A1

[4]

2. (a) The gradient of L_1

$= \frac{32-0}{24-8}$ (M1) for valid approach

$= 2$

The equation of L_1:

$y - 0 = 2(x-8)$ A1

$y = 2x - 16$

$2x - y - 16 = 0$ A1

[3]

(b) $2 \times -\frac{1}{-a} = -1$ (M1) for valid approach

$2 = -a$

$a = -2$ A1

[2]

3. (a) L.H.S.

$= (2n+1)^2 + (2n+3)^2 + (2n+5)^2$

$= 4n^2 + 4n + 1 + 4n^2 + 12n + 9 + 4n^2 + 20n + 25$ M1A1

$= 12n^2 + 36n + 35$

$= 12n^2 + 36n + 33 + 2$ M1

$= 3(4n^2 + 12n + 11) + 2$

$=$R.H.S. AG

[3]

(b) $2n+1$, $2n+3$ and $2n+5$ are three consecutive odd numbers. R1

$(2n+1)^2 + (2n+3)^2 + (2n+5)^2$
$= 3(4n^2 + 12n + 11) + 2$ A1

Also $3(4n^2 + 12n + 11)$ is a multiple of 3. R1

Thus, the sum of the squares of any three consecutive odd numbers is greater than a multiple of 3 by 2. AG

[3]

4. $f(x) = px^3 + qx^2 - 2x + 1$

$f'(x) = p(3x^2) + q(2x) - 2(1) + 0$ (A1) for correct derivatives

$f'(x) = 3px^2 + 2qx - 2$

$f'(1) = -1 \div -\frac{1}{15}$

$\therefore 3p(1)^2 + 2q(1) - 2 = 15$ (M1) for setting equation

$3p + 2q = 17$

$2q = 17 - 3p$ A1

$f^{-1}(41) = 2$

$\therefore f(2) = 41$ (M1) for valid approach

$p(2)^3 + q(2)^2 - 2(2) + 1 = 41$ A1

$8p + 4q - 3 = 41$

$\therefore 8p + 2(17 - 3p) - 3 = 41$ (M1) for substitution

$8p + 34 - 6p - 3 = 41$

$2p = 10$

$p = 5$ A1

$\therefore q = \frac{17 - 3(5)}{2}$

$q = 1$ A1

[8]

5.	(a)	$a = \frac{37-(-5)}{2}$	M1
		$a = 21$	A1
		$b = \frac{2\pi}{2(11-2)}$	M1
		$b = \frac{\pi}{9}$	A1
		$d = \frac{37+(-5)}{2}$	M1
		$d = 16$	
		$\therefore f(t) = 21\sin\frac{\pi}{9}(t+2.5)+16$	AG
			[5]
	(b)	The coordinates of P'	
		$=(3(2)+17, 37+8)$	A1
		$=(23, 45)$	A1
			[2]
6.	(a)	$g(x)$	
		$=3f(x-1)$	
		$=3(4(x-1)^4+3(x-1)^2-1)$	(A1) for substitution
		$=3(4(x^4-4x^3+6x^2-4x+1)+3(x^2-2x+1)-1)$	M1A1
		$=3(4x^4-16x^3+24x^2-16x+4+3x^2-6x+3-1)$	M1
		$=3(4x^4-16x^3+27x^2-22x+6)$	
		$=12x^4-48x^3+81x^2-66x+18$	A1
			[5]
	(b)	The sum of the roots	
		$=-\frac{-48}{12}$	M1
		$=4$	A1
			[2]

7. $1+f(|x|)\le|x|$

$1+\dfrac{2|x|^3-5|x|^2-37}{|x|+37}\le|x|$ M1

$\dfrac{2|x|^3-5|x|^2-37}{|x|+37}\le|x|-1$

$2|x|^3-5|x|^2-37\le(|x|-1)(|x|+37)$

$2|x|^3-5|x|^2-37\le|x|^2+36|x|-37$

$2|x|^3-6|x|^2-36|x|\le0$ (A1) for correct inequality

$2|x|\left(|x|^2-3|x|-18\right)\le0$

$|x|^2-3|x|-18\le0$ M1

$(|x|+3)(|x|-6)\le0$

$\therefore 0\le|x|\le6$ A1

$\therefore 1<x\le6$ A1

[5]

8. When $n=2$,

$\text{L.H.S.}=\binom{2}{2}$

$\text{L.H.S.}=1$

$\text{R.H.S.}=\dfrac{2(2+1)(2-1)}{6}$

$\text{R.H.S.}=1$

Thus, the statement is true when $n=2$. R1

Assume that the statement is true when $n=k$. M1

$$\binom{2}{2}+\binom{3}{2}+\ldots+\binom{k}{2}=\frac{k(k+1)(k-1)}{6}$$

When $n=k+1$,

$$\binom{2}{2}+\binom{3}{2}+\ldots+\binom{k}{2}+\binom{k+1}{2}$$

$$=\frac{k(k+1)(k-1)}{6}+\binom{k+1}{2}$$ M1A1

$$=\frac{k(k+1)(k-1)}{6}+\frac{(k+1)(k)}{2}$$ A1

$$=\frac{k(k+1)(k-1)}{6}+\frac{3k(k+1)}{6}$$

$$=\frac{k(k+1)}{6}(k-1+3)$$

$$=\frac{k(k+1)(k+2)}{6}$$

$$=\frac{(k+1)((k+1)+1)((k+1)-1)}{6}$$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$, $n\geq 2$. R1

[7]

9. (a) 1 A1

[1]

(b) $\int_1^a \frac{1}{e^2-1}e^{3-x}\mathrm{d}x=\frac{1}{2}$ A1

$\left[-\frac{1}{e^2-1}e^{3-x}\right]_1^a=\frac{1}{2}$ A1

$-\frac{1}{e^2-1}e^{3-a}-\left(-\frac{1}{e^2-1}e^2\right)=\frac{1}{2}$

$\frac{-e^{3-a}+e^2}{e^2-1}=\frac{1}{2}$ M1

$-e^{3-a}+e^2=\frac{1}{2}e^2-\frac{1}{2}$

$e^{3-a}=\frac{e^2+1}{2}$ A1

$3-a=\ln\left(\frac{e^2+1}{2}\right)$

$a=3-\ln\left(\frac{e^2+1}{2}\right)$

Thus, the median is $3-\ln\left(\frac{e^2+1}{2}\right)$. AG

[4]

Section B

10. (a) (i) $\{y : 0 \le y \le 1, y \in \mathbb{R}\}$ A2

(ii) $f(x) = 1$

$\therefore \cos^4 x = 1$

$\cos^2 x = -1$ (*Rejected*) or $\cos^2 x = 1$ (M1) for valid approach

$\cos x = -1$ or $\cos x = 1$

$x = \pi$ or $x = 0$, $x = 2\pi$ (A1) for correct values

Thus, there are 3 solutions. A1

[5]

(b) $f'(x) = (4\cos^3 x)(-\sin x)$ (A1) for chain rule

$f'(x) = -4\sin x \cos^3 x$ A1

[2]

(c) The total area of the regions

$= \int_0^{\pi} (\cos^4 x)(2\sin x)\mathrm{d}x$ (A1) for definite integral

Let $u = \cos x$
$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \Rightarrow (-1)\mathrm{d}u = \sin x\mathrm{d}x$
$x = \pi \Rightarrow u = \cos\pi = -1$
$x = 0 \Rightarrow u = \cos 0 = 1$

(A1) for substitution

$= \int_1^{-1} -2u^4 \mathrm{d}u$ M1A1

$= \left[-\frac{2}{5}u^5\right]_1^{-1}$ A1

$= -\frac{2}{5}(-1)^5 - \left(-\frac{2}{5}(1)^5\right)$ (M1) for substitution

$= \frac{4}{5}$ A1

[7]

11. (a) $\frac{dy}{dx}=h(x)\cdot(y+1)$

$\frac{1}{y+1}dy=\sin x dx$ (M1) for valid approach

$\int\frac{1}{y+1}dy=\int\sin x dx$ (A1) for correct approach

$\ln|y+1|=-\cos x+C$ A1

$y+1=e^{-\cos x+C}$ (M1) for valid approach

$y=e^{-\cos x+C}-1$ A1

$0=e^{-\cos 0+C}-1$ (M1) for substitution

$1=e^{-1+C}$

$-1+C=0$

$C=1$ (A1) for correct value

$\therefore y=e^{1-\cos x}-1$ A1

[8]

(b) $\frac{dy}{dx}=h(x)\sqrt{1-(h(x))^2}\cdot(y+1)$

$\frac{dy}{dx}=\sin x\sqrt{1-\sin^2 x}\cdot(y+1)$

$\frac{dy}{dx}=\sin x\cos x\cdot(y+1)$ A1

$\frac{dy}{dx}=\frac{\sin 2x\cdot(y+1)}{2}$

$\frac{dy}{dx}-\left(\frac{1}{2}\sin 2x\right)y=\frac{1}{2}\sin 2x$ A1

The integrating factor

$=e^{\int-\frac{1}{2}\sin 2x dx}$ M1

$=e^{\frac{1}{4}\cos 2x}$ A1

$\therefore e^{\frac{1}{4}\cos 2x}\frac{dy}{dx}-e^{\frac{1}{4}\cos 2x}\left(\frac{1}{2}\sin 2x\right)y=e^{\frac{1}{4}\cos 2x}\left(\frac{1}{2}\sin 2x\right)$ M1

$\therefore e^{\frac{1}{4}\cos 2x}\frac{dy}{dx}-\frac{1}{2}ye^{\frac{1}{4}\cos 2x}\sin 2x=\frac{1}{2}e^{\frac{1}{4}\cos 2x}\sin 2x$

$\frac{d}{dx}\left(ye^{\frac{1}{4}\cos 2x}\right)=\frac{1}{2}e^{\frac{1}{4}\cos 2x}\sin 2x$ A1

$ye^{\frac{1}{4}\cos 2x}=\int\frac{1}{2}e^{\frac{1}{4}\cos 2x}\sin 2x dx$

Let $u=\frac{1}{4}\cos 2x$. M1

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{4}(-\sin 2x)(2) \Rightarrow (-1)\mathrm{d}u=\frac{1}{2}\sin 2x\mathrm{d}x$$

$\therefore ye^{\frac{1}{4}\cos 2x}=\int -e^{u}\mathrm{d}u$ A1

$ye^{\frac{1}{4}\cos 2x}=-e^{u}+C$

$ye^{\frac{1}{4}\cos 2x}=-e^{\frac{1}{4}\cos 2x}+C$

$y=Ce^{-\frac{1}{4}\cos 2x}-1$ A1

$0=Ce^{-\frac{1}{4}\cos 2(0)}-1$ M1

$1=Ce^{-\frac{1}{4}}$

$C=e^{\frac{1}{4}}$ A1

$\therefore y=e^{\frac{1}{4}-\frac{1}{4}\cos 2x}-1$

$y=e^{\frac{1}{4}-\frac{1}{4}(1-2\sin^2 x)}-1$ A1

$y=e^{\frac{1}{2}\sin^2 x}-1$ AG

[12]

12. (a) $z^6+1=0$

$z^6=-1$

$z^6=\cos\pi+\mathrm{i}\sin\pi$ A1

$z=\cos\left(\dfrac{\pi+2k\pi}{6}\right)+\mathrm{i}\sin\left(\dfrac{\pi+2k\pi}{6}\right)$ M1

$(k=0,1,2,3,4,5)$

$z=\cos\dfrac{\pi}{6}+\mathrm{i}\sin\dfrac{\pi}{6}$, $z=\cos\dfrac{\pi}{2}+\mathrm{i}\sin\dfrac{\pi}{2}$,

$z=\cos\dfrac{5\pi}{6}+\mathrm{i}\sin\dfrac{5\pi}{6}$, $z=\cos\dfrac{7\pi}{6}+\mathrm{i}\sin\dfrac{7\pi}{6}$,

$z=\cos\dfrac{3\pi}{2}+\mathrm{i}\sin\dfrac{3\pi}{2}$ or $z=\cos\dfrac{11\pi}{6}+\mathrm{i}\sin\dfrac{11\pi}{6}$ A2

[4]

(b) z^6+1

$=z^6-z^4+z^2+z^4-z^2+1$ M1

$=z^2(z^4-z^2+1)+(z^4-z^2+1)$

$=(z^2+1)(z^4-z^2+1)$ A1

$z^4-z^2+1=0$

$\dfrac{z^6+1}{z^2+1}=0$, where $z^2\neq-1$ M1

$z=\cos\dfrac{\pi}{6}+\mathrm{i}\sin\dfrac{\pi}{6}$, $z=\cos\dfrac{\pi}{2}+\mathrm{i}\sin\dfrac{\pi}{2}$ (*Rejected*),

$z=\cos\dfrac{5\pi}{6}+\mathrm{i}\sin\dfrac{5\pi}{6}$, $z=\cos\dfrac{7\pi}{6}+\mathrm{i}\sin\dfrac{7\pi}{6}$,

$z=\cos\dfrac{3\pi}{2}+\mathrm{i}\sin\dfrac{3\pi}{2}$ (*Rejected*) or

$z=\cos\dfrac{11\pi}{6}+\mathrm{i}\sin\dfrac{11\pi}{6}$ A1

[4]

(c) (i) $(z-p)(z-q)=0$ M1

$z^2-(p+q)z+pq=0$

$p+q=\lambda^3+\lambda+\lambda^{11}+\lambda^9$

$p+q=\lambda^3+\lambda+\lambda^5(-1)+\lambda^3(-1)$ M1

$p+q=\lambda^3+\lambda-\lambda^5-\lambda^3$

$p+q=\lambda-\lambda^5$

$p+q=\lambda-\dfrac{-1}{\lambda}$ M1

$p+q=\lambda+\frac{1}{\lambda}$

$\therefore p+q=\sqrt{3}$ A1

$pq=(\lambda^3+\lambda)(\lambda^{11}+\lambda^9)$

$pq=\lambda^{14}+\lambda^{12}+\lambda^{12}+\lambda^{10}$ M1

$pq=\lambda^2(1)+1+1+\lambda^4(-1)$ M1

$pq=\lambda^2-\lambda^4+2$

$pq=\lambda^2-(\lambda^2-1)+2$

$pq=3$ A1

$\therefore z^2-\sqrt{3}z+3=0$ A1

(ii) $(z-(2p))(z-(2q))=0$ M1

$z^2-(2p+2q)z+(2p)(2q)=0$

$z^2-2(p+q)z+4pq=0$ A1

$z^2-2\sqrt{3}z+4(3)=0$ M1

$z^2-2\sqrt{3}z+12=0$ A1

[12]

AA HL Practice Set 1 Paper 2 Solution

Section A

1. (a)

$y = 3x + 7$

$\Rightarrow x = 3y + 7$ (A1) for correct approach

$3y = x - 7$

$y = \frac{x-7}{3}$

$\therefore f^{-1}(x) = \frac{x-7}{3}$ A1

[2]

(b)

$(f \circ g)(x)$

$= 3g(x) + 7$ (A1) for substitution

$= 3(2\sqrt{x}) + 7$

$= 6\sqrt{x} + 7$ A1

[2]

(c)

$(f \circ g)(529)$

$= 6\sqrt{529} + 7$ (M1) for substitution

$= 145$ A1

[2]

2. (a) The volume

$=\frac{1}{3}\pi r^2 h$ (M1) for valid approach

$=\frac{1}{3}\pi(18)^2(18)$

$=6107.256119$ (A1) for correct value

$=6110$

$=6.11\times10^3\text{ cm}^3$ A1

[3]

(b) $V=27\left(\frac{2}{3}\pi R^3\right)$ (M1) for setting equation

$16(6107.256119)=18\pi R^3$ (A1) for substitution

$R^3=1728$

$R=12$ A1

The ratio

$=18:12$

$=3:2$ A1

[4]

3. (a) $r=\frac{5.4}{4.5}$ (M1) for valid approach

$r=1.2$ A1

[2]

(b) $S_{12}=\frac{4.5(1.2^{12}-1)}{1.2-1}$ (A1) for substitution

$S_{12}=178.1122601$

$S_{12}=178$ A1

[2]

(c) $u_n<678$

$4.5\cdot1.2^{n-1}<678$

$4.5\cdot1.2^{n-1}-678<0$ (M1) for valid approach

By considering the graph of $y=4.5\cdot1.2^{n-1}-678$,

$n<28.50673$. A1

Thus, the greatest value of n is 28. A1

[3]

4. (a) $20P_1 - 17P_0 = 0$

$\therefore 20(P_0 e^{k(1)}) - 17P_0 = 0$ A1

$20e^k - 17 = 0$

$e^k = 0.85$ M1

$k = \ln 0.85$ AG

[2]

(b) $\frac{P_t}{P_0} \le 0.5$

$\therefore \frac{P_0 e^{(\ln 0.85)t}}{P_0} \le 0.5$ (A1) for correct inequality

$e^{(\ln 0.85)t} \le 0.5$ (A1) for correct approach

$(\ln 0.85)t \le \ln 0.5$

$(\ln 0.85)t - \ln 0.5 \le 0$ A1

By considering the graph of

$y = (\ln 0.85)t - \ln 0.5$, $t \ge 4.2650243$. (M1) for valid approach

Thus, the least number of whole years is 43. A1

[5]

5. (a) $\text{AB}^2 = r^2 + r^2 - 2(r)(r)\cos 2\alpha$ A1

$\text{AB}^2 = 2r^2 - 2r^2 \cos 2\alpha$

$\text{AB} = \sqrt{2r^2 - 2r^2 \cos 2\alpha}$ A1

$\text{AB} = \sqrt{2r^2(1 - \cos 2\alpha)}$

$\text{AB} = r\sqrt{2(1 - \cos 2\alpha)}$ AG

[2]

(b) The arc length ACB

$= (r)(2\alpha)$ A1

$= 2r\alpha$

$\therefore P$

$= 2r\alpha + r\sqrt{2(1 - \cos 2\alpha)}$ M1

$= 2r\alpha + r\sqrt{2(1 - (1 - 2\sin^2 \alpha))}$ A1

$= 2r\alpha + r\sqrt{2(2\sin^2 \alpha)}$ A1

$= 2r\alpha + r\sqrt{4\sin^2 \alpha}$

$= 2r\alpha + 2r\sin\alpha$ A1

$= 2r(\alpha + \sin\alpha)$ AG

[5]

6. By using row operations, the system

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 5 & 7 & 1 & 2 \\ 32 & 24 & -17 & 5 \end{array}\right)$$ is reduced to $$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{11}{8} & -\frac{1}{8} \\ 0 & 1 & \frac{9}{8} & \frac{3}{8} \\ 0 & 0 & 0 & 0 \end{array}\right).$$ (M1) for valid approach

$y+\frac{9}{8}z=\frac{3}{8}$

$y=\frac{3}{8}-\frac{9}{8}z$ A1

$x-\frac{11}{8}z=-\frac{1}{8}$

$x=-\frac{1}{8}+\frac{11}{8}z$ A1

Let $z=t$.

$x=-\frac{1}{8}+\frac{11}{8}t$, $y=\frac{3}{8}-\frac{9}{8}t$

Thus, the vector equation of the line of intersection is

$$\mathbf{r}=\begin{pmatrix} -\frac{1}{8} \\ \frac{3}{8} \\ 0 \end{pmatrix}+t\begin{pmatrix} \frac{11}{8} \\ -\frac{9}{8} \\ 1 \end{pmatrix}.$$ A2

[5]

7. (a) $\frac{1-x}{1+ax}$

$= (1-x)(1+ax)^{-1}$

$= (1-x)\left(1+(-1)(ax)+\frac{(-1)(-2)}{2!}(ax)^2+\cdots\right)$ M1A1

$= (1-x)(1-ax+a^2x^2+\cdots)$

$= 1-ax+a^2x^2-x+ax^2-a^2x^3+\cdots$

$= 1+(-a-1)x+(a^2+a)x^2+\cdots$ A1

[3]

(b) (i) $1+(a^2+a)=21$ (A1) for correct equation

$a^2+a-20=0$

$(a+5)(a-4)=0$

$a=-5$ (*Rejected*) or $a=4$ A1

(ii) -5 A1

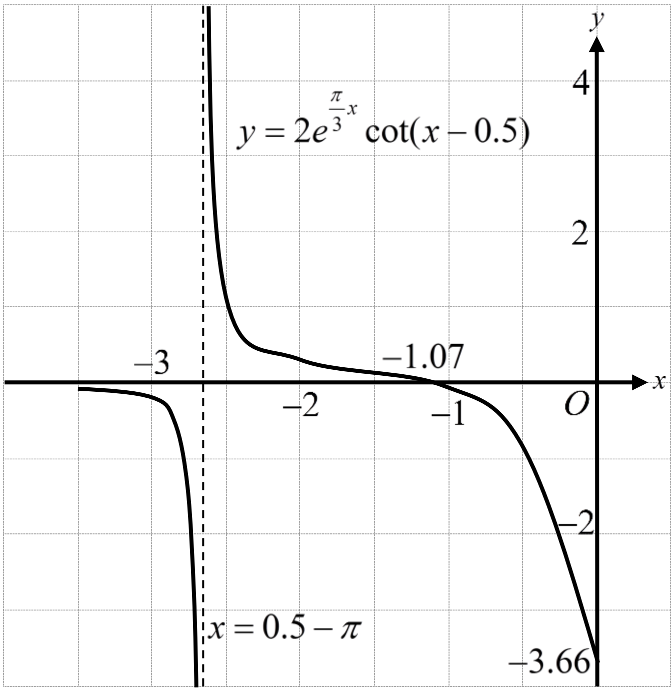
[3]

8. (a) For correct shape A1

For correct asymptote A1

For correct intercepts A1

[3]



(b) $0.0442 \le k \le 3.66$ A2

[2]

9. $y=4^{-x}$

$\log_4 y=-x$

$x=-\log_4 y$ (A1) for correct approach

$y=4^{-0}$

$y=1$ (A1) for correct value

$-\log_4 y=-\frac{1}{32}(y-24)^2$ (M1) for setting equation

$\frac{1}{32}(y-24)^2-\log_4 y=0$

By considering the graph of $x=\frac{1}{32}(y-24)^2-\log_4 y$,

$y=16$. (A1) for correct value

$0=-\frac{1}{32}(y-24)^2$

$0=(y-24)^2$

$y=24$ (A1) for correct value

The area of R

$=-\int_1^{16}(-\log_4 y)\mathrm{d}y-\int_{16}^{24}-\frac{1}{32}(y-24)^2\mathrm{d}y$ A1

$=26.51312053$

$=26.5$ A1

[7]

Section B

10. (a) The required probability

$= P(T \leq 24)$ — (M1) for valid approach

$= 0.9452007106$

$= 0.945$ — A1

[2]

(b) $P(U \leq 48) = 0.99494$

$P\left(Z \leq \frac{48-\mu}{7}\right) = 0.99494$ — (M1) for standardization

$\frac{48-\mu}{7} = 2.571701859$ — A1

$48 - \mu = 18.00191301$

$\mu = 29.99808699$

$\mu = 30.0$ — A1

[3]

(c) The required probability

$= P(U \leq 36)$ — R1

$= 0.8043925789$ — A1

Thus, for all school buses departing at $8:24 \text{ am}$, 80.439% of them will arrive at school on time. — AG

[2]

(d) The required probability

$= 1 - P(T \leq 12)P(U \leq 48)$

$- P(12 < T \leq 24)P(U \leq 36)$ — M1A1

$= 1 - (0.2118553337)(0.99494)$

$- (0.7333453769)(0.80439)$ — (A2) for correct values

$= 0.1993209666$

$= 0.199$ — A1

[5]

(e) The expected number

$= (20)(0.1993209666)$ — (A1) for correct formula

$= 3.986419331$

$= 3.99$ — A1

[2]

11. (a) Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the normal vectors of the planes π_1 and π_2 respectively.

$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 3 \\ k \end{pmatrix}$, $\mathbf{n}_2 = \begin{pmatrix} 4 \\ -3 \\ k \end{pmatrix}$ (A1) for correct values

$\mathbf{n}_1 \times \mathbf{n}_2 = \beta \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} (3)(k)-(k)(-3) \\ (k)(4)-(4)(k) \\ (4)(-3)-(3)(4) \end{pmatrix} = \beta \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$, where β is a constant. (A1) for substitution

$\begin{pmatrix} 6k \\ 0 \\ -24 \end{pmatrix} = \beta \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$\therefore \frac{6k}{-24} = \frac{-3}{1}$ A1

$k = 12$ A1

[4]

(b) (i) $a = 6$, $b = 8$, $c = 2$, $\alpha = -6$ A4

(ii) Let O be the origin.

The volume of the pyramid $\mathrm{A'ABC}$

$= \frac{1}{3}\left(\frac{(\mathrm{A'A})(\mathrm{OB})}{2}\right)(\mathrm{OC})$ (M1) for valid approach

$= \frac{1}{3}\left(\frac{(6-(-6))(8)}{2}\right)(2)$ A1

$= 32$ A1

[7]

(c) (i) $\overrightarrow{AC'} = -6\mathbf{i} - 2\mathbf{k}$

$\overrightarrow{AC'} \cdot (-\mathbf{i}) = \left|\overrightarrow{AC'}\right| \left|-\mathbf{i}\right| \cos C'\hat{A}A'$ (M1) for valid approach

$(-6\mathbf{i} - 2\mathbf{k}) \cdot (-\mathbf{i})$
$= (\sqrt{(-6)^2 + (-2)^2})(1)\cos C'\hat{A}A'$ (A1) for substitution

$(-6)(-1) + (-2)(0) = \sqrt{40}\cos C'\hat{A}A'$

$\cos C'\hat{A}A' = \dfrac{6}{\sqrt{40}}$

$C'\hat{A}A' = 18.43494882^\circ$

$C'\hat{A}A' = 18.4^\circ$ A1

(ii) $\because C'A' = C'A$

$\therefore C'\hat{A}'A = 18.43494882^\circ$ (A1) for correct approach

$A\hat{C}'A' + 18.43494882^\circ + 18.43494882^\circ = 180^\circ$

$A\hat{C}'A' = 143.1301024^\circ$

$A\hat{C}'A' = 143^\circ$ A1

[5]

(d) The vector equation of L:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 4 \\ 3 \\ 12 \end{pmatrix}$$

$\begin{cases} x = 4s \\ y = 3s \\ z = 2 + 12s \end{cases}$ (A1) for correct approach

$\therefore 4(4s) - 3(3s) + 12(2 + 12s) = -24$ (A1) for substitution

$151s = -48$

$s = -\dfrac{48}{151}$

$\begin{cases} x = 4\left(-\dfrac{48}{151}\right) = -1.271523179 \\ y = 3\left(-\dfrac{48}{151}\right) = -0.9536423841 \\ z = 2 + 12\left(-\dfrac{48}{151}\right) = -1.814569536 \end{cases}$ M1

Thus, the coordinates of Q are
$(-1.2715, -0.9536, -1.8146)$. A1

[4]

12. (a) (i) $f'(x)=\left(\frac{1}{x^2+1}\right)(2x)$

$f'(x)=\frac{2x}{x^2+1}$ A1

$f''(x)=\frac{(x^2+1)(2)-(2x)(2x)}{(x^2+1)^2}$ (M1) for valid approach

$f''(x)=\frac{2x^2+2-4x^2}{(x^2+1)^2}$

$f''(x)=\frac{2-2x^2}{(x^2+1)^2}$ A1

$f^{(3)}(x)=\frac{(x^2+1)^2(-4x)-(2-2x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$ (M1) for valid approach

$f^{(3)}(x)=\frac{-4x^3-4x-8x+8x^3}{(x^2+1)^3}$

$f^{(3)}(x)=\frac{4x^3-12x}{(x^2+1)^3}$ A1

(ii) $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)$

$+\frac{x^3}{3!}f^{(3)}(0)+\frac{x^4}{4!}f^{(4)}(0)+\cdots$

$f(x)=\ln(0^2+1)+x\left(\frac{2(0)}{0^2+1}\right)$

$+\frac{x^2}{2}\left(\frac{2-2(0)^2}{(0^2+1)^2}\right)+\frac{x^3}{6}\left(\frac{4(0)^3-12(0)}{(0^2+1)^3}\right)$ M2

$+\frac{x^4}{24}\left(-\frac{12(0^4-6(0)^2+1)}{(0^2+1)^4}\right)+\cdots$

$f(x)=0+x(0)+\frac{x^2}{2}(2)$

$+\frac{x^3}{6}(0)+\frac{x^4}{24}(-12)+\cdots$ A2

$f(x)=x^2-\frac{1}{2}x^4+\cdots$ A1

[10]

(b) $\sin x = x - \frac{x^3}{3!} + \cdots$

$\ln\left((x^2+1)^{\sin x}\right)$

$= \sin x \ln(x^2+1)$ (A1) for correct approach

$= \left(x - \frac{x^3}{6} + \cdots\right)\left(x^2 - \frac{1}{2}x^4 + \cdots\right)$ M1A1

$= x^3 - \frac{1}{2}x^5 - \frac{1}{6}x^5 + \cdots$ (M1) for valid approach

$= x^3 - \frac{2}{3}x^5 + \cdots$ A1

[5]

(c) The approximate value of the volume

$= \int_{0.7}^{1.3} \pi\left(y\sqrt{\ln\left((y^2+1)^{\sin y}\right)}\right)^2 \mathrm{d}y$ (M1) for valid approach

$= \int_{0.7}^{1.3} \pi y^2 \ln\left((y^2+1)^{\sin y}\right)\mathrm{d}y$

$\approx \int_{0.7}^{1.3} \pi y^2\left(y^3 - \frac{2}{3}y^5\right)\mathrm{d}y$ A1

$\approx \int_{0.7}^{1.3} \pi\left(y^5 - \frac{2}{3}y^7\right)\mathrm{d}y$ (M1) for valid approach

≈ 0.3452245902

≈ 0.345 A1

[4]

AA HL Practice Set 1 Paper 3 Solution

1. (a) (i) $\dfrac{2\pi}{3}$ A1

(ii) A_1

$= \pi(1)^2 - 3\left(\dfrac{1}{2}(1)^2 \sin\dfrac{2\pi}{3}\right)$ M1A1

$= \pi - 3\left(\dfrac{1}{2}\sin\dfrac{2\pi}{3}\right)$

$= \pi - \dfrac{3}{2}\sin\dfrac{2\pi}{3}$ A1

$= \dfrac{3}{2}\left(\dfrac{2\pi}{3} - \sin\dfrac{2\pi}{3}\right)$

$= \left(\dfrac{1}{2}+1\right)\left(\dfrac{2\pi}{3} - \sin\dfrac{2\pi}{3}\right)$ AG

[4]

(b) (i) $\dfrac{\pi}{3}$ A1

(ii) $\dfrac{1}{2}\sin\dfrac{\pi}{3}$ A1

(iii) A_2

$= \pi(1)^2 - \dfrac{1}{2}\sin\dfrac{2\pi}{3} - 4\left(\dfrac{1}{2}\sin\dfrac{\pi}{3}\right)$ M1A1

$= \pi - \dfrac{1}{2}\sin\dfrac{2\pi}{3} - 2\sin\dfrac{\pi}{3}$

$= \dfrac{\pi}{3} - \dfrac{1}{2}\sin\dfrac{2\pi}{3} + \dfrac{2\pi}{3} - 2\sin\dfrac{\pi}{3}$ M1

$= \dfrac{1}{2}\left(\dfrac{2\pi}{3} - \sin\dfrac{2\pi}{3}\right) + 2\left(\dfrac{\pi}{3} - \sin\dfrac{\pi}{3}\right)$ AG

[5]

(c) (i) $Q_2\hat{O}Q$

$= \frac{2\pi}{3} \div 3$ (M1) for valid approach

$= \frac{2\pi}{9}$ A1

(ii) A_3

$= \pi(1)^2 - \frac{1}{2}\sin\frac{2\pi}{3} - 6\left(\frac{1}{2}\sin\frac{2\pi}{9}\right)$ M1A1

$= \pi - \frac{1}{2}\sin\frac{2\pi}{3} - 3\sin\frac{2\pi}{9}$

$= \frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} + \frac{2\pi}{3} - 3\sin\frac{2\pi}{9}$ M1

$= \frac{1}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) + 3\left(\frac{2\pi}{9} - \sin\frac{2\pi}{9}\right)$ A1

[6]

(d) (i) A_n

$= \pi(1)^2 - \frac{1}{2}\sin\frac{2\pi}{3} - 2n\left(\frac{1}{2}\sin\left(\frac{2\pi}{3} \div n\right)\right)$ M1A1

$= \pi - \frac{1}{2}\sin\frac{2\pi}{3} - n\sin\frac{2\pi}{3n}$

$= \frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} + \frac{2\pi}{3} - n\sin\frac{2\pi}{3n}$ M1

$= \frac{1}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) + n\left(\frac{2\pi}{3n} - \sin\frac{2\pi}{3n}\right)$ A1

$\therefore f(n) = \frac{2\pi}{3n} - \sin\frac{2\pi}{3n}$ A1

(ii) $f(n)$ represents the double of the area of the segment of the sector POQ_1. A1

[6]

(e) $\lim_{n\to\infty} f(n)$

$= \lim_{n\to\infty}\left(\frac{2\pi}{3n} - \sin\frac{2\pi}{3n}\right)$

$= \frac{2\pi}{3}\lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty}\sin\frac{2\pi}{3n}$ M1

$= \frac{2\pi}{3}\lim_{n\to\infty}\frac{1}{n} - \sin\left(\frac{2\pi}{3}\lim_{n\to\infty}\frac{1}{n}\right)$

$= \frac{2\pi}{3}(0) - \sin\left(\frac{2\pi}{3}(0)\right)$

$= 0$ A1

[2]

(f) (i) $\frac{3}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$ A2

(ii) The maximum possible value of v

$= \lim_{n\to\infty} A_n$ M1

$= \lim_{n\to\infty}\left(\frac{1}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) + n\cdot f(n)\right)$

$= \frac{1}{2}\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$ A1

[4]

2. (a) (i) $w^2 - w + 1 = 0$

$$w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$ (A1) for substitution

$$w = \frac{1 \pm \sqrt{-3}}{2}$$

$$w = \frac{1 \pm \sqrt{3}\mathrm{i}}{2}$$

$$w = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$$

$w = \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}$ or

$w = \cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)$ A2

(ii) $u^4 - u^2 + 1 = 0$

$(u^2)^2 - u^2 + 1 = 0$ M1

$u^2 = \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}$ or

$u^2 = \cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)$

$u = \cos\left(\frac{\frac{\pi}{3} + 2\pi k}{2}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{3} + 2\pi k}{2}\right)$ or

$u = \cos\left(\frac{-\frac{\pi}{3} + 2\pi k}{2}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{3} + 2\pi k}{2}\right)$

$(k = 0, 1)$ A1

$u = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}$, $u = \cos\frac{7\pi}{6} + \mathrm{i}\sin\frac{7\pi}{6}$,

$u = \cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)$ or

$u = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}$ A2

Thus, the required roots are

$\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)$,

$\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)$, $\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}$ and

$\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}$. AG

(iii) $z^{2n}-z^{n}+1=0$

$(z^{n})^{2}-z^{n}+1=0$

$z^{n}=\cos\frac{\pi}{3}+\text{i}\sin\frac{\pi}{3}$ or

$z^{n}=\cos\left(-\frac{\pi}{3}\right)+\text{i}\sin\left(-\frac{\pi}{3}\right)$

$z=\cos\left(\frac{\frac{\pi}{3}+2\pi k}{n}\right)+\text{i}\sin\left(\frac{\frac{\pi}{3}+2\pi k}{n}\right)$ or

$z=\cos\left(\frac{-\frac{\pi}{3}+2\pi k}{n}\right)+\text{i}\sin\left(\frac{-\frac{\pi}{3}+2\pi k}{n}\right)$

$(k=0,1,2,\cdots,n-1))$ A1

$z=\cos\left(\frac{\pi}{3n}+\frac{2\pi}{n}k\right)+\text{i}\sin\left(\frac{\pi}{3n}+\frac{2\pi}{n}k\right)$ or

$z=\cos\left(-\frac{\pi}{3n}+\frac{2\pi}{n}k\right)+\text{i}\sin\left(-\frac{\pi}{3n}+\frac{2\pi}{n}k\right)$

$(k=0,1,2,\cdots,n-1))$

$z=\cos\frac{\pi+6\pi k}{3n}+\text{i}\sin\frac{\pi+6\pi k}{3n}$ or

$z=\cos\frac{-\pi+6\pi k}{3n}+\text{i}\sin\frac{-\pi+6\pi k}{3n}$

$(k=0,1,2,\cdots,n-1))$ A1

Thus, the required roots are

$\cos\left(-\frac{\pi}{3n}\right)+\text{i}\sin\left(-\frac{\pi}{3n}\right)$, $\cos\frac{\pi}{3n}+\text{i}\sin\frac{\pi}{3n}$,

$\cos\frac{5\pi}{3n}+\text{i}\sin\frac{5\pi}{3n}$, $\cos\frac{7\pi}{3n}+\text{i}\sin\frac{7\pi}{3n}$, $\cdots$,

$\cos\frac{(6n-7)\pi}{3n}+\text{i}\sin\frac{(6n-7)\pi}{3n}$ and

$\cos\frac{(6n-5)\pi}{3n}+\text{i}\sin\frac{(6n-5)\pi}{3n}$. A3

[12]

(b) (i) $(z-(\cos\theta+\mathrm{i}\sin\theta))(z-(\cos(-\theta)+\mathrm{i}\sin(-\theta)))$

$=(z-\cos\theta-\mathrm{i}\sin\theta)(z-\cos\theta+\mathrm{i}\sin\theta)$

$=z^2-z\cos\theta+\mathrm{i}z\sin\theta-z\cos\theta+\cos^2\theta$ M1

$-\mathrm{i}\sin\theta\cos\theta-\mathrm{i}z\sin\theta+\mathrm{i}\sin\theta\cos\theta+\sin^2\theta$

$=z^2-2z\cos\theta+1$ A1

(ii) u^4-u^2+1

$=\left(u-\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)\right)$

$\left(u-\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)\right)$ M1A1

$\left(u-\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)\right)$

$\left(u-\left(\cos\left(-\frac{5\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)\right)$

$=\left(u^2-2u\cos\frac{\pi}{6}+1\right)\left(u^2-2u\cos\frac{5\pi}{6}+1\right)$ AG

(iii) The roots of the equation $z^6-z^3+1=0$

are $\operatorname{cis}\frac{\pi}{9}$, $\operatorname{cis}\left(-\frac{\pi}{9}\right)$, $\operatorname{cis}\frac{5\pi}{9}$, $\operatorname{cis}\left(-\frac{5\pi}{9}\right)$,

$\operatorname{cis}\frac{7\pi}{9}$ and $\operatorname{cis}\left(-\frac{7\pi}{9}\right)$. (A1) for correct values

z^6-z^3+1

$=\left(z-\operatorname{cis}\frac{\pi}{9}\right)\left(z-\operatorname{cis}\left(-\frac{\pi}{9}\right)\right)\left(z-\operatorname{cis}\frac{5\pi}{9}\right)$

$\left(z-\operatorname{cis}\left(-\frac{5\pi}{9}\right)\right)\left(z-\operatorname{cis}\frac{7\pi}{9}\right)$ A1

$\left(z-\operatorname{cis}\left(-\frac{7\pi}{9}\right)\right)$

$=\left(z^2-2z\cos\frac{\pi}{9}+1\right)\left(z^2-2z\cos\frac{5\pi}{9}+1\right)$ A1

$\left(z^2-2z\cos\frac{7\pi}{9}+1\right)$

(iv) $z^{2n} - z^{n} + 1 = 0$

$$= \left(z^2 - 2z\cos\frac{\pi}{3n} + 1 \right)\left(z^2 - 2z\cos\frac{5\pi}{3n} + 1 \right)$$

$$\left(z^2 - 2z\cos\frac{7\pi}{3n} + 1 \right)\cdots$$

$$\left(z^2 - 2z\cos\left(\pi - \frac{5\pi}{3n} \right) + 1 \right)$$ A2

$$\left(z^2 - 2z\cos\left(\pi - \frac{\pi}{3n} \right) + 1 \right)$$

[9]

(c) $u^4 - u^2 + 1 = \left(u^2 - 2u\cos\frac{\pi}{6} + 1 \right)\left(u^2 - 2u\cos\frac{5\pi}{6} + 1 \right)$

When $u = \mathrm{i}$,

$\mathrm{i}^4 - \mathrm{i}^2 + 1 = \left(\mathrm{i}^2 - 2\mathrm{i}\cos\frac{\pi}{6} + 1 \right)\left(\mathrm{i}^2 - 2\mathrm{i}\cos\frac{5\pi}{6} + 1 \right)$ M1

$1 - (-1) + 1 = \left(-1 - 2\mathrm{i}\cos\frac{\pi}{6} + 1 \right)\left(-1 - 2\mathrm{i}\cos\frac{5\pi}{6} + 1 \right)$ A1

$3 = \left(-2\mathrm{i}\cos\frac{\pi}{6} \right)\left(-2\mathrm{i}\cos\frac{5\pi}{6} \right)$

$3 = 4\mathrm{i}^2\cos\frac{\pi}{6}\cos\frac{5\pi}{6}$ A1

$3 = -4\cos\frac{\pi}{6}\cos\frac{5\pi}{6}$

$\cos\frac{\pi}{6}\cos\frac{5\pi}{6} = -\frac{3}{4}$ AG

[3]

(d)

$$z^6 - z^3 + 1 = \left(z^2 - 2z\cos\frac{\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{7\pi}{9} + 1\right)$$

When $z = \text{i}$,

$$\text{i}^6 - \text{i}^3 + 1 = \left(\text{i}^2 - 2\text{i}\cos\frac{\pi}{9} + 1\right)\left(\text{i}^2 - 2\text{i}\cos\frac{5\pi}{9} + 1\right)\left(\text{i}^2 - 2\text{i}\cos\frac{7\pi}{9} + 1\right)$$ (M1) for valid approach

$$-1 - (-\text{i}) + 1 = \left(-1 - 2\text{i}\cos\frac{\pi}{9} + 1\right)\left(-1 - 2\text{i}\cos\frac{5\pi}{9} + 1\right)\left(-1 - 2\text{i}\cos\frac{7\pi}{9} + 1\right)$$ A1

$$\text{i} = \left(-2\text{i}\cos\frac{\pi}{9}\right)\left(-2\text{i}\cos\frac{5\pi}{9}\right)\left(-2\text{i}\cos\frac{7\pi}{9}\right)$$

$$\text{i} = -8\text{i}^3\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}$$ A1

$$\text{i} = 8\text{i}\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}$$

$$\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}$$ A1

[4]

AA HL Practice Set 2 Paper 1 Solution

Section A

1. (a) (i) 7 A1

(ii) 1 A1

[2]

(b) $(f \circ g)(x)$

$= (g(x))^2$ (A1) for substitution

$= (3-4x)^2$

$= 9-24x+16x^2$ A1

[2]

(c) $y = 3-4x$

$\Rightarrow x = 3-4y$ (A1) for correct approach

$4y = 3-x$

$y = \dfrac{3-x}{4}$

$\therefore g^{-1}(x) = \dfrac{3-x}{4}$ A1

[2]

2. (a) R.H.S.

$$=\frac{1\times 49}{1\times 49}+\frac{2\times 7}{7\times 7}+\frac{5}{49}$$ M1

$$=\frac{49+14+5}{49}$$ A1

$$=\frac{68}{49}=\text{L.H.S.}$$

$$\therefore \frac{68}{49}=1+\frac{2}{7}+\frac{5}{49}$$ AG

[2]

(b) R.H.S.

$$=\frac{1\times(m+2)^2}{1\times(m+2)^2}+\frac{2\times(m+2)}{(m+2)\times(m+2)}+\frac{5}{(m+2)^2}$$ M1

$$=\frac{(m^2+4m+4)+(2m+4)+5}{(m+2)^2}$$ M1A1

$$=\frac{m^2+6m+9+4}{(m+2)^2}$$

$$=\frac{(m+3)^2+4}{(m+2)^2}=\text{L.H.S.}$$

$$\therefore \frac{(m+3)^2+4}{(m+2)^2}\equiv 1+\frac{2}{m+2}+\frac{5}{(m+2)^2} \text{ for } m\neq -2$$ AG

[3]

3. $P(2)=0$

$a(2)^3+b(2)^2-10(2)+24=0$ (M1) for factor theorem

$4b=-4-8a$

$b=-1-2a$ A1

$P(-3)=0$

$a(-3)^3+b(-3)^2-10(-3)+24=0$

$-27a+9b+30+24=0$

$\therefore -27a+9(-1-2a)+30+24=0$ (M1) for substitution

$-27a-9-18a+30+24=0$

$-45a=-45$

$a=1$ A1

$b=-1-2(1)$

$b=-3$ A1

[5]

4. (a) The discriminant of $f(x)$

$= b^2 - 4ac$

$= (8-p)^2 - 4\left(1 + 2p - \frac{3}{8}p^2\right)(-2)$ M1A1

$= 64 - 16p + p^2 + 8 + 16p - 3p^2$ A1

$= 72 - 2p^2$ AG

[3]

(b) $f(x) = 0$ has two equal roots

$\therefore 72 - 2p^2 = 0$ (M1) for setting equation

$2p^2 = 72$

$p^2 = 36$

$p = -6$ or $p = 6$ A2

[3]

(c) $p = 6$

$\therefore \left(1 + 2(6) - \frac{3}{8}(6)^2\right)x^2 + (8-6)x - 2 = 0$ (M1) for setting equation

$-\frac{1}{2}x^2 + 2x - 2 = 0$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2$ A1

[2]

5. $9\log_{27}(x+1) = 1 + \log_3(3 + x + x^2)$

$\frac{9\log_3(x+1)}{\log_3 27} = \log_3 3 + \log_3(3 + x + x^2)$ (M1)(A1) for change of base

$\frac{9\log_3(x+1)}{3} = \log_3 3(3 + x + x^2)$ (A1) for correct approach

$3\log_3(x+1) = \log_3 3(3 + x + x^2)$

$\log_3(x+1)^3 = \log_3 3(3 + x + x^2)$ A1

$\therefore (x+1)^3 = 3(3 + x + x^2)$ M1

$x^3 + 3x^2 + 3x + 1 = 9 + 3x + 3x^2$

$x^3 = 8$ A1

$x = \sqrt[3]{8}$

$x = 2$ A1

[7]

6. (a) $r = \dfrac{20\cos^4\alpha}{30\cos^2\alpha}$ (M1) for valid approach

$r = \dfrac{2}{3}\cos^2\alpha$ A1

[2]

(b) $\pi \le \alpha \le \dfrac{4}{3}\pi$

$\therefore \cos\pi \le \cos\alpha \le \cos\dfrac{4}{3}\pi$ (M1) for valid approach

$-1 \le \cos\alpha \le -\dfrac{1}{2}$

$\dfrac{1}{4} \le \cos^2\alpha \le 1$

$\dfrac{1}{6} \le \dfrac{2}{3}\cos^2\alpha \le \dfrac{2}{3}$

$\therefore \dfrac{1}{6} \le r \le \dfrac{2}{3}$ A1

[2]

(c) $S_\infty = \dfrac{30\cos^2\alpha}{1-\dfrac{2}{3}\cos^2\alpha}$ A1

$S_\infty = \dfrac{30\cos^2\alpha}{\sin^2\alpha+\cos^2\alpha-\dfrac{2}{3}\cos^2\alpha}$ M1

$S_\infty = \dfrac{30\cos^2\alpha}{\sin^2\alpha+\dfrac{1}{3}\cos^2\alpha}$ A1

$S_\infty = \dfrac{30}{\tan^2\alpha+\dfrac{1}{3}}$ A1

$S_\infty = \dfrac{90}{3\tan^2\alpha+1}$ AG

[4]

7. When $n=1$,

$\text{L.H.S.}=1^2$

$\text{L.H.S.}=1$

$\text{R.H.S.}=\frac{4}{3}(1)^3$

$\text{R.H.S.}=\frac{4}{3}$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$1^2+2^2+\ldots+k^2\le\frac{4}{3}k^3$

When $n=k+1$,

$1^2+2^2+\ldots+k^2+(k+1)^2$

$\le\frac{4}{3}k^3+(k+1)^2$ M1A1

$=\frac{4k^3+3(k^2+2k+1)}{3}$ A1

$=\frac{4k^3+3k^2+6k+3}{3}$

$\le\frac{4k^3+12k^2+12k+4}{3}$ A1

$=\frac{4(k^3+3k^2+3k+1)}{3}$

$=\frac{4}{3}(k+1)^3$

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

8. $\lim_{x\to 0}\frac{1-e^{x^2}}{1-\sec x}$

$=\lim_{x\to 0}\frac{0-(e^{x^2})(2x)}{-\sec x\tan x}\left(\because \frac{0}{0}\right)$ M1A2

$=\lim_{x\to 0}\frac{2xe^{x^2}}{\sec x\tan x}$

$=\lim_{x\to 0}\frac{(2)(e^{x^2})+(2x)(e^{x^2})(2x)}{(\sec x\tan x)(\tan x)+(\sec x)(\sec^2 x)}\left(\because \frac{0}{0}\right)$ A2

$=\lim_{x\to 0}\frac{2e^{x^2}+4x^2e^{x^2}}{\sec x\tan^2 x+\sec^3 x}$

$=\frac{2e^0+4(0)^2e^0}{\sec 0\tan^2 0+\sec^3 0}$ M1

$=\frac{2+0}{0+1}$

$=2$ A1

[7]

9. (a) $-\pi a$ A1

[1]

(b) $\int_{-\pi a}^{a}|x|\,\mathrm{d}x=1$ A1

$\int_{-\pi a}^{0}-x\mathrm{d}x+\int_{0}^{a}x\mathrm{d}x=1$

$\left[-\frac{1}{2}x^2\right]_{-\pi a}^{0}+\left[\frac{1}{2}x^2\right]_{0}^{a}=1$ A1

$\left(0-\left(-\frac{1}{2}\pi^2a^2\right)\right)+\left(\frac{1}{2}a^2-0\right)=1$

$\frac{1}{2}\pi^2a^2+\frac{1}{2}a^2=1$ M1

$a^2(\pi^2+1)=2$

$a^2=\frac{2}{\pi^2+1}$ A1

$a=-\sqrt{\frac{2}{\pi^2+1}}$ (*Rejected*) or $a=\sqrt{\frac{2}{\pi^2+1}}$

Thus, $a=\sqrt{\frac{2}{\pi^2+1}}$. AG

[4]

Section B

10. (a)

$2r+h=20$	(A1) for correct approach
$2r=20-h$	
$r=10-\frac{1}{2}h$	A1
	[2]

(b)

$V=\pi r^2 h$	
$V=\pi\left(10-\frac{1}{2}h\right)^2 h$	(A1) for substitution
$V=100\pi h-10\pi h^2+\frac{1}{4}\pi h^3$	A1
	[2]

(c)

$Q=(3)(2\pi rh)+(4)(\pi r^2)$	M1A1
$Q=6\pi\left(10-\frac{1}{2}h\right)h+4\pi\left(10-\frac{1}{2}h\right)^2$	M1
$Q=60\pi h-3\pi h^2+400\pi-40\pi h+\pi h^2$	A1
$Q=400\pi+20\pi h-2\pi h^2$	
$Q=2\pi(200+10h-h^2)$	AG
	[4]

(d)

$\frac{\mathrm{d}Q}{\mathrm{d}h}=2\pi(0+10(1)-2h)$	(A1) for correct derivatives
$\frac{\mathrm{d}Q}{\mathrm{d}h}=4\pi(5-h)$	A1
$\frac{\mathrm{d}Q}{\mathrm{d}h}=0$	(M1) for setting equation
$\therefore 4\pi(5-h)=0$	A1
$h=5$	A1
The maximum value of Q	
$=2\pi(200+10(5)-(5)^2)$	(M1) for substitution
$=450\pi$	A1
	[7]

11. (a) $\overrightarrow{BD} = \begin{pmatrix} -9 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix}$

$\overrightarrow{BD} = \begin{pmatrix} -9 \\ -9 \\ 9 \end{pmatrix}$ A1

The vector equation of BD:

$\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix} + t\begin{pmatrix} -9 \\ -9 \\ 9 \end{pmatrix}$

$\begin{cases} x = -9t \\ y = 9 - 9t \\ z = -9 + 9t \end{cases}$ A1

$\overrightarrow{CE} = \begin{pmatrix} -9t \\ 9 - 9t \\ -9 + 9t \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -9 \end{pmatrix}$

$\overrightarrow{CE} = \begin{pmatrix} -9t \\ 9 - 9t \\ 9t \end{pmatrix}$ A1

$\overrightarrow{CE} \cdot \overrightarrow{BD} = 0$

$\therefore (-9t)(-9) + (9 - 9t)(-9) + (9t)(9) = 0$ M1

$81t - 81 + 81t + 81t = 0$

$243t = 81$

$t = \frac{1}{3}$ A1

$\therefore \begin{cases} x = -9\left(\frac{1}{3}\right) = -3 \\ y = 9 - 9\left(\frac{1}{3}\right) = 6 \\ z = -9 + 9\left(\frac{1}{3}\right) = -6 \end{cases}$ M1

Therefore, the coordinates of E are $(-3, 6, -6)$.AG

[6]

(b) $\vec{BA} = \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix}$

$\vec{BA} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$ (A1) for correct values

$\vec{BC} = \begin{pmatrix} 0 \\ 0 \\ -9 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix}$

$\vec{BC} = \begin{pmatrix} 0 \\ -9 \\ 0 \end{pmatrix}$ (A1) for correct values

$\mathbf{n}_1 = \vec{BA} \times \vec{BD}$ (M1) for valid approach

$\mathbf{n}_1 = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix} \times \begin{pmatrix} -9 \\ -9 \\ 9 \end{pmatrix}$

$\mathbf{n}_1 = \begin{pmatrix} (0)(9) - (9)(-9) \\ (9)(-9) - (0)(9) \\ (0)(-9) - (0)(-9) \end{pmatrix}$

$\mathbf{n}_1 = \begin{pmatrix} 81 \\ -81 \\ 0 \end{pmatrix}$ A1

$\mathbf{n}_2 = \vec{BC} \times \vec{BD}$ (M1) for valid approach

$\mathbf{n}_2 = \begin{pmatrix} 0 \\ -9 \\ 0 \end{pmatrix} \times \begin{pmatrix} -9 \\ -9 \\ 9 \end{pmatrix}$

$\mathbf{n}_2 = \begin{pmatrix} (-9)(9) - (0)(-9) \\ (0)(-9) - (0)(9) \\ (0)(-9) - (-9)(-9) \end{pmatrix}$

$\mathbf{n}_2 = \begin{pmatrix} -81 \\ 0 \\ -81 \end{pmatrix}$ A1

$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1||\mathbf{n}_2| \cos\theta$ (M1) for valid approach

$(81)(-81)+(-81)(0)+(0)(-81)$

$=(\sqrt{81^2+(-81)^2})(\sqrt{(-81)^2+(-81)^2})\cos\theta$ A1

$-81^2=2(81)^2\cos\theta$

$\cos\theta=-\frac{1}{2}$

$\theta=120^\circ$ A1

[9]

(c) The area of OABC

$=(\text{OA})(\text{OC})$

$=(9)(9)$

$=81$ (A1) for correct value

$\therefore \frac{1}{3}(81)(\text{OD})+\frac{1}{3}(81)(\text{OF})=783$ (M1) for setting equation

$\frac{1}{3}(81)(9)+\frac{1}{3}(81)(\text{OF})=783$

$27\text{OF}=540$

$\text{OF}=20$ A1

$\therefore \text{DF}=9+20$

$\text{DF}=29$ A1

[4]

12. (a) $\frac{da}{dt} - 2a^2 = 50$

$\frac{da}{dt} = 2a^2 + 50$

$\frac{da}{dt} = 2(a^2 + 25)$

$\frac{1}{a^2+25}da = 2dt$ (M1) for valid approach

$\int \frac{1}{a^2+25}da = \int 2dt$ (A1) for correct approach

$\int \frac{1}{a^2+5^2}da = \int 2dt$

$\frac{1}{5}\arctan\frac{a}{5} = 2t + C$ A1

$\arctan\frac{a}{5} = 10t + C$

$\frac{a}{5} = \tan(10t + C)$

$a = 5\tan(10t + C)$ A1

$5 = 5\tan(10(0) + C)$ (M1) for substitution

$1 = \tan C$

$\tan C = \tan\frac{\pi}{4}$

$C = \frac{\pi}{4}$ (A1) for correct value

$\therefore a = 5\tan\left(10t + \frac{\pi}{4}\right)$ A1

[7]

(b) $\frac{dv}{dt} = 5\tan\left(10t + \frac{\pi}{4}\right)$

$dv = \frac{5\sin\left(10t + \frac{\pi}{4}\right)}{\cos\left(10t + \frac{\pi}{4}\right)}dt$

$\int dv = \int \frac{5\sin\left(10t + \frac{\pi}{4}\right)}{\cos\left(10t + \frac{\pi}{4}\right)}dt$ (A1) for correct approach

Let $u = \cos\left(10t + \frac{\pi}{4}\right)$. (M1) for substitution

$$\frac{du}{dt} = -10\sin\left(10t+\frac{\pi}{4}\right) \Rightarrow 5\sin\left(10t+\frac{\pi}{4}\right)dt = -\frac{1}{2}du$$

$\therefore \int dv = -\frac{1}{2}\int \frac{1}{u}du$ (A1) for correct working

$v = -\frac{1}{2}\ln u + D$

$v = -\frac{1}{2}\ln\left|\cos\left(10t+\frac{\pi}{4}\right)\right| + D$ A1

$\ln 2^{\frac{1}{4}} = -\frac{1}{2}\ln\left|\cos\left(10(0)+\frac{\pi}{4}\right)\right| + D$ (M1) for substitution

$\frac{1}{4}\ln 2 = -\frac{1}{2}\ln\frac{\sqrt{2}}{2} + D$

$\frac{1}{4}\ln 2 = \frac{1}{4}\ln 2 + D$

$D = 0$ (A1) for correct value

$\therefore v = -\frac{1}{2}\ln\left|\cos\left(10t+\frac{\pi}{4}\right)\right|$ A1

[7]

(c) $v = -\frac{1}{2}\ln\left|\cos\left(10t+\frac{\pi}{4}\right)\right|$

$v = \frac{1}{2}\ln\left|\frac{1}{\cos\left(10t+\frac{\pi}{4}\right)}\right|$ M1

$v = \frac{1}{2}\ln\left|\sec\left(10t+\frac{\pi}{4}\right)\right|$ A1

$v = \frac{1}{4}\ln\left(\sec^2\left(10t+\frac{\pi}{4}\right)\right)$

$v = \frac{1}{4}\ln\left(\tan^2\left(10t+\frac{\pi}{4}\right)+1\right)$ A1

$\therefore v = \frac{1}{4}\ln\left(\left(\frac{a}{5}\right)^2+1\right)$ A1

$v = \frac{1}{4}\ln\left(\frac{a^2}{25}+1\right)$

$v = \frac{1}{4}\ln\left(\frac{a^2+25}{25}\right)$ AG

[4]

AA HL Practice Set 2 Paper 2 Solution

Section A

1.

$$\left(kx-\frac{4}{x}\right)^8=(kx)^8+\binom{8}{1}(kx)^7\left(-\frac{4}{x}\right)+\binom{8}{2}(kx)^6\left(-\frac{4}{x}\right)^2$$

$$+\binom{8}{3}(kx)^5\left(-\frac{4}{x}\right)^3+\binom{8}{4}(kx)^4\left(-\frac{4}{x}\right)^4+\ldots$$

(M1)(A1) for correct approach

$$\left(kx-\frac{4}{x}\right)^8=k^8x^8+8k^7x^7\left(-\frac{4}{x}\right)+28k^6x^6\left(\frac{16}{x^2}\right)$$

$$+56k^5x^5\left(-\frac{64}{x^3}\right)+70k^4x^4\left(\frac{256}{x^4}\right)+\ldots$$

(A1) for simplification

$$\left(kx-\frac{4}{x}\right)^8=k^8x^8-32k^7x^6+448k^6x^4$$

$$-3584k^5x^2+17920k^4+\ldots$$

A1

$\therefore 448k^6:17920k^4=9:40$ A1

$$\frac{448k^6}{17920k^4}=\frac{9}{40}$$

$$\frac{k^2}{40}=\frac{9}{40}$$

$k=-3$ or $k=3$ (*Rejected*) A1

[6]

2.	(a)	$A = 2\pi r^2 + 2\pi rh + 2\pi r^2$	(M2) for setting equation
		$135\pi = 4\pi r^2 + 2\pi r(3.5)$	(A1) for substitution
		$135 = 4r^2 + 7r$	
		$4r^2 + 7r - 135 = 0$	(M1) for quadratic equation
		$(4r + 27)(r - 5) = 0$	
		$4r + 27 = 0$ or $r - 5 = 0$	
		$r = -\dfrac{27}{4}$ (*Rejected*) or $r = 5\text{ mm}$	A1

[5]

	(b)	The volume	
		$= \dfrac{4}{3}\pi r^3 + \pi r^2 h$	(M1) for valid approach
		$= \dfrac{4}{3}\pi(5)^3 + \pi(5)^2(3.5)$	
		$= 798.4881328\text{ mm}^3$	
		$= 798\text{ mm}^3$	A1

[2]

3.	(a)	(i)	$\cos \hat{ABC} = \dfrac{r^2 + (1.75r)^2 - (1.5r)^2}{2(r)(1.75r)}$	M1A1
			$\cos \hat{ABC} = \dfrac{1.8125r^2}{3.5r^2}$	A1
			$\cos \hat{ABC} = \dfrac{29}{56}$	AG
		(ii)	$\hat{ABC} = 1.026452178\text{ rad}$	
			$\hat{ABC} = 1.03\text{ rad}$	A1

[4]

	(b)	$\dfrac{1}{2}(\text{BC})^2(\hat{ABC}) = 9.89$	(M1) for setting equation
		$\dfrac{1}{2}r^2(\pi - 1.026452178) = 9.89$	(A1) for substitution
		$r^2 = 9.35162474$	
		$r = 3.058042632$	
		$r = 3.06$	A1

[3]

4. $X \sim \mathrm{B}\left(5, \dfrac{2p}{p+2p+10}\right)$ (R1) for correct distribution

The standard deviation of X

$= \sqrt{5\left(\dfrac{2p}{3p+10}\right)\left(1-\dfrac{2p}{3p+10}\right)}$ (A1) for substitution

$= \sqrt{5\left(\dfrac{2p}{3p+10}\right)\left(\dfrac{p+10}{3p+10}\right)}$

$\therefore \sqrt{5\left(\dfrac{2p}{3p+10}\right)\left(\dfrac{p+10}{3p+10}\right)} > \dfrac{11}{10}$ (M1) for valid approach

$5\left(\dfrac{2p}{3p+10}\right)\left(\dfrac{p+10}{3p+10}\right) > \dfrac{121}{100}$ M1

$5\left(\dfrac{2p}{3p+10}\right)\left(\dfrac{p+10}{3p+10}\right) - \dfrac{121}{100} > 0$ A1

By considering the graph of

$y = 5\left(\dfrac{2p}{3p+10}\right)\left(\dfrac{p+10}{3p+10}\right) - \dfrac{121}{100}$,

$5.3435147 < p < 25.443002$.

Thus, the greatest value of p is 25. A1

[6]

5. $v = \int (8-8t)\,\mathrm{d}t$ (M1) for indefinite integral

$v = 8t - 8\left(\dfrac{1}{2}t^2\right) + C$ A1

$v = 8t - 4t^2 + C$

The initial velocity

$= 8(0) - 4(0)^2 + C$ (M1) for valid approach

$= C$

The difference between the velocities is $4\ \mathrm{ms}^{-1}$

$\therefore 8t - 4t^2 + C = C + 4$ or $\therefore 8t - 4t^2 + C = C - 4$ (A1) for correct approach

$4t^2 - 8t + 4 = 0$ or $4t^2 - 8t - 4 = 0$ A2

$4(t-1)^2 = 0$ or $t = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(-4)}}{2(4)}$

$t = 1$ or $t = 2.414213562$, $t = -0.4142135624$ (*Rejected*)

$\therefore m = 1$ or $m = 2.41$ A2

[8]

6. (a) By using row operations, the system

$$\left(\begin{array}{ccc|c} 2 & -1 & -3 & 3 \\ 1 & -4 & -6 & -17 \\ 3 & 1 & 2 & 21 \end{array}\right) \text{ is reduced to}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right).$$ (M1) for valid approach

Thus, the coordinates of P are $(5, 4, 1)$. A3

[4]

(b) $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -4 \\ -6 \end{pmatrix}$ A2

[2]

7. (a) $\{x : -5 \le x \le 1\}$ A2

[2]

(b) $f(x) = 2-(x-1)^2$

$y = 2-(x-1)^2$

$\Rightarrow x = 2-(y-1)^2$ (M1) for swapping variables

$(y-1)^2 = 2-x$

$y-1 = \sqrt{2-x}$ (*Rejected*) or $y-1 = -\sqrt{2-x}$ A1

$y = -\sqrt{2-x}+1$

$\therefore f^{-1}(x) = -\sqrt{2-x}+1$ A1

[3]

(c) $(g^{-1} \circ f^{-1})(x) = \dfrac{x}{3}$

$f^{-1}(x) = g\left(\dfrac{x}{3}\right)$ M1

$g\left(\dfrac{x}{3}\right) = -\sqrt{2-x}+1$

$g\left(3\left(\dfrac{x}{3}\right)\right) = -\sqrt{2-3x}+1$ A1

$\therefore g(x) = -\sqrt{2-3x}+1$ A1

[3]

8. $\frac{2\pi}{B} = 2(4-0)$

$\frac{2\pi}{B} = 8$

$B = \frac{\pi}{4}$ A1

$5+\pi = A\sec\frac{\pi}{4}(0)+C$

$5+\pi = A+C$

$C = 5+\pi-A$

$5-\pi = A\sec\frac{\pi}{4}(4)+C$

$\therefore 5-\pi = A(-1)+5+\pi-A$ (M1) for substitution

$-2\pi = -2A$

$A = \pi$ A1

$C = 5+\pi-\pi$

$C = 5$ A1

[4]

9. (a) The total number of possible ways

$= \frac{14!}{14\times 2}$ (A2) for correct formula

$= 3113510400$ A1

[3]

(b) The number of possible ways

$= 3113510400 - \frac{2!\times 13!}{13\times 2}$ (A2) for correct formula

$= 2634508800$ A1

[3]

Section B

10.	(a)	$P(L > 59.2) = 0.12$	(M1) for valid approach
		$P\left(Z > \dfrac{59.2 - \mu}{3.5}\right) = 0.12$	(A1) for correct approach
		$\dfrac{59.2 - \mu}{3.5} = 1.174986791$	A1
		$59.2 - \mu = 4.11245377$	
		$\mu = 55.08754623$	
		$\mu = 55.1$	A1
			[4]
	(b)	$P(L < q) = 0.55$	
		$P\left(Z < \dfrac{q - 55.08754623}{3.5}\right) = 0.55$	(A1) for correct approach
		$\dfrac{q - 55.08754623}{3.5} = 0.1256613375$	
		$q - 55.08754623 = 0.4398146813$	
		$q = 55.52736091$	A1
		$\therefore q = 55.5$	A1
			[3]
	(c) (i)	$X \sim B(10, 0.55)$	(R1) for correct distribution
		$E(X) = (10)(0.55)$	(A1) for substitution
		$E(X) = 5.5$	A1
	(ii)	$P(X > 5) = 1 - P(X \le 5)$	(M1) for valid approach
		$P(X > 5) = 1 - 0.4955954083$	A1
		$P(X > 5) = 0.5044045917$	
		$P(X > 5) = 0.504$	A1
			[6]
	(d)	$m\left(\dfrac{55\%}{55\% + 33\%}\right)(0.8) + m\left(\dfrac{33\%}{55\% + 33\%}\right)(1.1)$	(M1)(A1) for correct approach
		$= (949)(1000)$	
		$0.5m + 0.4125m = 949000$	A1
		$0.9125m = 949000$	
		$m = 1040000$	A1
			[4]

11. (a) When $0 \le t \le 1$,

$s(t) = \int \pi t \mathrm{d}t$	(M1) for valid approach
$s(t) = \frac{\pi}{2}t^2 + C$	(A1) for correct value
$s(0) = -\frac{\pi}{2}$	
$\therefore \frac{\pi}{2}(0)^2 + C = -\frac{\pi}{2}$	
$C = -\frac{\pi}{2}$	
$s(1) = \frac{\pi}{2}(1)^2 - \frac{\pi}{2}$	
$s(1) = 0$	(A1) for correct value
When $1 < t \le 5$,	
$s(t) = \int \pi e^{1-t} \mathrm{d}t$	(M1) for valid approach
$s(t) = -\pi e^{1-t} + D$	(A1) for correct value
$s(1) = 0$	
$\therefore -\pi e^{1-1} + D = 0$	
$D = \pi$	(A1) for correct value
$s(5) = -\pi e^{1-5} + \pi$	
$s(5) = -\pi e^{-4} + \pi$	
$s(5) = \frac{\pi}{e^4}(e^4 - 1)$	(A1) for correct value
$\therefore s(t) = \begin{cases} \frac{\pi}{2}t^2 - \frac{\pi}{2} & 0 \le t \le 1 \\ -\pi e^{1-t} + \pi & 1 < t \le 5 \\ \frac{\pi}{e^4}(e^4 - 1) & t > 5 \end{cases}$	A1

[8]

(b) $a(t)=\begin{cases}\pi(1) & 0\le t\le 1\\ \pi e^{1-t}(-1) & 1<t\le 5\\ 0 & \text{otherwise}\end{cases}$ (M1) for valid approach

$a(t)=\begin{cases}\pi & 0\le t\le 1\\ -\pi e^{1-t} & 1<t\le 5\\ 0 & \text{otherwise}\end{cases}$ (A1) for correct values

$a(t)<-3$

$-\pi e^{1-t}<-3$ (M1) for setting inequality

$3-\pi e^{1-t}<0$

By considering the graph of $y=3-\pi e^{1-t}$,

$t<1.0461176$.

$\therefore 1<t<1.05$ A1

[4]

(c) (i) $\frac{ds}{dt}=\pi e^{1-t}$

$\frac{dv}{dt}=-\pi e^{1-t}$

$\therefore \frac{ds}{dv}$

$=\frac{ds}{dt}\div\frac{dv}{dt}$ M1

$=\frac{\pi e^{1-t}}{-\pi e^{1-t}}$ A1

$=-1$ AG

(ii) $\frac{dt}{dv}$

$=1\div\frac{dv}{dt}$ M1

$=\frac{1}{-\pi e^{1-t}}$ A1

$=-\frac{1}{\pi}e^{t-1}$ AG

[4]

12. (a) $|z|$

$= \left| \dfrac{\frac{4}{5}e^{i\theta}}{2} \right|$ (A1) for correct approach

$= \left| \dfrac{2}{5}e^{i\theta} \right|$

$= \dfrac{2}{5}\left| e^{i\theta} \right|$

$= \dfrac{2}{5}(1)$

$= \dfrac{2}{5}$ A1

[2]

(b) $\dfrac{2}{1-\frac{2}{5}e^{i\theta}}$ A2

[2]

(c) (i) $\dfrac{2}{1-\frac{2}{5}e^{i\theta}}$

$= \dfrac{10}{5-2e^{i\theta}}$

$= \dfrac{10(5-2e^{i(-\theta)})}{(5-2e^{i\theta})(5-2e^{i(-\theta)})}$ M1A1

$= \dfrac{50-20e^{i(-\theta)}}{25-10e^{i(-\theta)}-10e^{i\theta}+4}$ M1

$= \dfrac{50-20e^{i(-\theta)}}{29-10(e^{i(-\theta)}+e^{i\theta})}$

$= \dfrac{50-20(\cos(-\theta)+i\sin(-\theta))}{29-10(\cos(-\theta)+i\sin(-\theta)+\cos\theta+i\sin\theta)}$ A1

$= \dfrac{50-20(\cos\theta-i\sin\theta)}{29-10(\cos\theta-i\sin\theta+\cos\theta+i\sin\theta)}$

$= \dfrac{50-20\cos\theta+20i\sin\theta}{29-10(2\cos\theta)}$ M1

$= \dfrac{(50-20\cos\theta)+i(20\sin\theta)}{29-20\cos\theta}$ A1

$\frac{4}{5}\sin\theta+\frac{8}{25}\sin 2\theta+\frac{16}{125}\sin 3\theta+\cdots$

$=\frac{20\sin\theta}{29-20\cos\theta}$ M1A1

$\therefore \sin\theta+\frac{2}{5}\sin 2\theta+\frac{4}{25}\sin 3\theta+\cdots$

$=\frac{25\sin\theta}{29-20\cos\theta}$ AG

(ii) $2+\frac{4}{5}\cos\theta+\frac{8}{25}\cos 2\theta+\cdots=\frac{50-20\cos\theta}{29-20\cos\theta}$ M1A1

$\frac{4}{5}\cos\theta+\frac{8}{25}\cos 2\theta+\frac{16}{125}\cos 3\theta+\cdots$

$=\frac{50-20\cos\theta}{29-20\cos\theta}-2$ M1

$=\frac{50-20\cos\theta}{29-20\cos\theta}-\frac{2(29-20\cos\theta)}{29-20\cos\theta}$

$=\frac{50-20\cos\theta}{29-20\cos\theta}-\frac{58-40\cos\theta}{29-20\cos\theta}$ A1

$=\frac{50-20\cos\theta-58+40\cos\theta}{29-20\cos\theta}$ M1

$=\frac{-8+20\cos\theta}{29-20\cos\theta}$ A1

$\therefore \cos\theta+\frac{2}{5}\cos 2\theta+\frac{4}{25}\cos 3\theta+\cdots$

$=\frac{-10+25\cos\theta}{29-20\cos\theta}$ A1

$=\frac{5(-2+5\cos\theta)}{29-20\cos\theta}$ AG

[15]

AA HL Practice Set 2 Paper 3 Solution

1. (a) arc P_1B

$= \frac{1}{4}\pi(1)^2$ (M1) for valid approach

$= \frac{1}{4}\pi$ A1

[2]

(b) (i) 1 A1

(ii) $\sqrt{2}$ A1

[2]

(c) (i) $R(3)$

$= 3\left(\frac{1}{2}(\text{OA})(\text{OP}_1)\sin \text{A}\hat{\text{O}}\text{P}_1\right)$ (M1) for valid approach

$= 3\left(\frac{1}{2}(1)(1)\sin\frac{180^\circ}{3}\right)$ (A1) for substitution

$= \frac{3}{2}\sin 60^\circ$ A1

(ii) $\text{AP}_1 = \text{OP}_1$ as AOP_1 is an equilateral triangle. R1

[4]

(d) $R(4)$

$= 4\left(\frac{1}{2}(\text{OA})(\text{OP}_1)\sin \text{A}\hat{\text{O}}\text{P}_1\right)$ M1

$= 4\left(\frac{1}{2}(1)(1)\sin\frac{180^\circ}{4}\right)$ A1

$= 2\sin 45^\circ$ AG

[2]

(e) $AP_1^2 = OA^2 + OP_1^2 - 2(OA)(OP_1)\cos A\hat{O}P_1$ M1

$L(4)^2 = 1^2 + 1^2 - 2(1)(1)\cos 45^\circ$ A1

$L(4)^2 = 2 - 2\left(\frac{\sqrt{2}}{2}\right)$

$L(4)^2 = 2 - \sqrt{2}$ A1

$\because L(4)^4 - 4L(4)^2 + 2$

$= (2-\sqrt{2})^2 - 4(2-\sqrt{2}) + 2$ M1

$= 4 - 4\sqrt{2} + 2 - 8 + 4\sqrt{2} + 2$ M1

$= 0$

Thus, the exact value of $L(4)$ satisfies the equation $x^4 - 4x^2 + 2 = 0$. AG

[5]

(f) (i) $R(n)$

$= n\left(\frac{1}{2}(OA)(OP_1)\sin A\hat{O}P_1\right)$ M1

$= n\left(\frac{1}{2}(1)(1)\sin\frac{180^\circ}{n}\right)$ A1

$= \frac{n}{2}\sin\frac{180^\circ}{n}$ A1

(ii) $\frac{1}{2}\pi$ A1

[4]

(g) (i) $AP_1^2 = OA^2 + OP_1^2 - 2(OA)(OP_1)\cos A\hat{O}P_1$ M1

$L(n)^2 = 1^2 + 1^2 - 2(1)(1)\cos\frac{180^\circ}{n}$ A1

$L(n)^2 = 2 - 2\cos\frac{180^\circ}{n}$

$L(n) = \sqrt{2 - 2\cos\frac{180^\circ}{n}}$ AG

(ii) $\frac{L(n)}{R(n)}$

$= \frac{\sqrt{2-2\cos\frac{180^\circ}{n}}}{\frac{n}{2}\sin\frac{180^\circ}{n}}$

$= \frac{\sqrt{2-2\left(1-2\sin^2\frac{90^\circ}{n}\right)}}{\frac{n}{2}\left(2\sin\frac{90^\circ}{n}\cos\frac{90^\circ}{n}\right)}$ A2

$= \frac{\sqrt{4\sin^2\frac{90^\circ}{n}}}{n\sin\frac{90^\circ}{n}\cos\frac{90^\circ}{n}}$ M1

$= \frac{2\sin\frac{90^\circ}{n}}{n\sin\frac{90^\circ}{n}\cos\frac{90^\circ}{n}}$ M1

$= \frac{2}{n\cos\frac{90^\circ}{n}}$

$= \frac{2}{n}\sec\frac{90^\circ}{n}$ AG

[6]

(h) $\frac{L(n)}{R(n)} < \frac{1}{\pi^\pi}$

$\therefore \frac{2}{n}\sec\frac{90^\circ}{n} < \frac{1}{\pi^\pi}$

$\frac{2}{n}\sec\frac{90^\circ}{n} - \frac{1}{\pi^\pi} < 0$ (A1) for correct inequality

By considering the graph of $y = \frac{2}{n}\sec\frac{90^\circ}{n} - \frac{1}{\pi^\pi}$,

$n > 72.941232$.

Thus, the least value of n is 73. A1

[2]

2. (a) $f'(x)$

$= (e^x)(1-x)^n + (e^x)(n)(1-x)^{n-1}(-1)$ A1

$= e^x(1-x)^{n-1}\left[(1-x)+n(-1)\right]$

$= e^x(1-x)^{n-1}(1-x-n)$ A1

$e^x > 0$, $(1-x)^{n-1} > 0$ and $1-x-n < 0$ for $n > 0$. R1

$\therefore f'(x) < 0$

Thus, $f(x)$ is decreasing in $0 < x < 1$ for $n > 0$. AG

[3]

(b) $f(0) = 1$ and $f(1) = 0$. R1

Also, $f(x)$ is decreasing in $0 < x < 1$.

Therefore, the area under the graph of $f(x)$ is positive, and is smaller than the area of the square of length 1. R1

Thus, $0 < I(n) < 1$ for $n > 0$. AG

[2]

(c) (i) $I(0)$

$= \int_0^1 e^x(1-x)^0 \mathrm{d}x$ M1

$= \int_0^1 e^x \mathrm{d}x$

$= \left[e^x\right]_0^1$ A1

$= e^1 - e^0$

$= e - 1$ AG

(ii) $I(1)$

$= \int_0^1 e^x (1-x)^1 \mathrm{d}x$

Let $\theta = e^x$. (M1) for valid approach

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = e^x$

$\therefore I(1)$

$= \int_0^1 (1-x) \cdot \frac{\mathrm{d}(e^x)}{\mathrm{d}x} \mathrm{d}x$

$= \left[(1-x)e^x \right]_0^1 - \int_0^1 e^x \cdot \frac{\mathrm{d}(1-x)}{\mathrm{d}x} \mathrm{d}x$ A1

$= \left[(1-x)e^x \right]_0^1 - \int_0^1 e^x (-1) \mathrm{d}x$ A1

$= \left[(1-x)e^x \right]_0^1 + \int_0^1 e^x \mathrm{d}x$

$= \left[(1-x)e^x \right]_0^1 + e - 1$ A1

$= ((1-1)e^1 - (1-0)e^0) + e - 1$

$= (0-1) + e - 1$

$= e - 2$ A1

(iii) $I(2)$

$= \int_0^1 e^x (1-x)^2 \mathrm{d}x$

Let $\theta = e^x$. (M1) for valid approach

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = e^x$

$\therefore I(2)$

$= \int_0^1 (1-x)^2 \cdot \frac{\mathrm{d}(e^x)}{\mathrm{d}x} \mathrm{d}x$

$= \left[(1-x)^2 e^x \right]_0^1 - \int_0^1 e^x \cdot \frac{\mathrm{d}((1-x)^2)}{\mathrm{d}x} \mathrm{d}x$ A1

$= \left[(1-x)^2 e^x \right]_0^1 - \int_0^1 e^x \cdot 2(1-x)(-1) \mathrm{d}x$ A1

$= \left[(1-x)^2 e^x \right]_0^1 + 2\int_0^1 e^x (1-x) \mathrm{d}x$

$= \left[(1-x)^2 e^x \right]_0^1 + 2I(1)$ A1

$= \left[(1-x)^2 e^x \right]_0^1 + 2(e-2)$

$= ((1-1)^2 e^1 - (1-0)^2 e^0) + 2(e-2)$

$= (0-1) + 2e - 4$

$= 2e - 5$ A1

[12]

(d) $I(n)$

$=\int_0^1 e^x(1-x)^n\,\mathrm{d}x$

Let $\theta=e^x$. M1

$\frac{\mathrm{d}\theta}{\mathrm{d}x}=e^x$

$\therefore I(n)$

$=\int_0^1(1-x)^n\cdot\frac{\mathrm{d}(e^x)}{\mathrm{d}x}\mathrm{d}x$

$=\left[(1-x)^n e^x\right]_0^1-\int_0^1 e^x\cdot\frac{\mathrm{d}((1-x)^n)}{\mathrm{d}x}\mathrm{d}x$ A1

$=\left[(1-x)^n e^x\right]_0^1-\int_0^1 e^x\cdot n(1-x)^{n-1}(-1)\mathrm{d}x$ A1

$=\left[(1-x)^n e^x\right]_0^1+n\int_0^1 e^x(1-x)^{n-1}\mathrm{d}x$

$=\left[(1-x)^n e^x\right]_0^1+nI(n-1)$ A1

$=((1-1)^n e^1-(1-0)^n e^0)+nI(n-1)$

$=-1+nI(n-1)$ A1

Thus, $I(n)=nI(n-1)-1$ for $n>0$. AG

[5]

(e) $I(n)$

$=nI(n-1)-1$

$=n((n-1)I(n-2)-1)-1$ M1

$=n(n-1)I(n-2)-n-1$

$=n(n-1)((n-2)I(n-3)-1)-n-1$ M1

$=n(n-1)(n-2)I(n-3)-n(n-1)-n-1$

$=\cdots$

$=n(n-1)(n-2)\cdots(2)(1)I(0)$

$-n(n-1)(n-2)\cdots(2)$ A1

$-\cdots-n(n-1)(n-2)-n(n-1)-n-1$

$=n!\left[\begin{array}{l}I(0)-\frac{1}{1!}-\cdots\\-\frac{1}{(n-3)!}-\frac{1}{(n-2)!}-\frac{1}{(n-1)!}-\frac{1}{n!}\end{array}\right]$ M1A1

$=n!\left[e-1-\left(\frac{1}{1!}+\cdots+\frac{1}{(n-2)!}+\frac{1}{(n-1)!}+\frac{1}{n!}\right)\right]$

$\therefore I(n)=n!\left[e-1-\sum_{r=1}^{n}\frac{1}{r!}\right]$ AG

[5]

(f) e A1

[1]

AA HL Practice Set 3 Paper 1 Solution

Section A

1. (a) The common difference

$=95-100$ (M1) for valid approach

$=-5$ A1

[2]

(b) The fifteenth term

$=100+(15-1)(-5)$ (A1) for substitution

$=30$ A1

[2]

(c) The sum of the first fifteen terms

$=\frac{15}{2}\left[2(100)+(15-1)(-5)\right]$ (A1) for substitution

$=975$ A1

[2]

2. (a) The gradient of L_1 is 2. A1

The y-intercept of L_1 is -20. A1

[2]

(b) The gradient of L_2 is $-\frac{1}{2}$. (A1) for correct value

The equation of L_2:

$y-(-20)=-\frac{1}{2}(x-0)$ A1

$y+20=-\frac{1}{2}x$

$2y+40=-x$

$x+2y+40=0$ A1

[3]

3. (a) (i) 4 A1

(ii) $\frac{1}{3}$ A1

(iii) -1 A1

[3]

(b) $\log_{27} x+\frac{8}{3}=\log_4 256+\log_{125} 5+\log_\pi \frac{1}{\pi}$

$\log_{27} x+\frac{8}{3}=4+\frac{1}{3}-1$ (M1) for substitution

$\log_{27} x=\frac{2}{3}$

$x=27^{\frac{2}{3}}$ (A1) for correct approach

$x=(3^3)^{\frac{2}{3}}$

$x=3^2$

$x=9$ A1

[3]

4. $\left(1-\frac{3}{4}x\right)^n(1+2nx)^3$

$=\left(1+\binom{n}{1}\left(-\frac{3}{4}x\right)+\cdots\right)\left(1+\binom{3}{1}(2nx)+\cdots\right)$ (M1) for valid expansion

$=\left(1+(n)\left(-\frac{3}{4}x\right)+\cdots\right)(1+(3)(2nx)+\cdots)$ (A1) for correct approach

$=\left(1-\frac{3}{4}nx+\cdots\right)(1+6nx+\cdots)$ A2

The coefficient of x

$=(1)(6n)+\left(-\frac{3}{4}n\right)(1)$ (A1) for correct approach

$=\frac{21}{4}n$

$\therefore \frac{21}{4}n=\frac{105}{4}$ (M1) for setting equation

$n=5$ A1

[7]

5. $-3\sqrt{3} \le f(x) \le 3\sqrt{3}$

$-3\sqrt{3} \le 6\sin 2x \le 3\sqrt{3}$

$-\frac{\sqrt{3}}{2} \le \sin 2x \le \frac{\sqrt{3}}{2}$ A1

$\therefore \sin\left(-\frac{\pi}{3}\right) \le \sin 2x \le \sin\frac{\pi}{3}$,

$\sin\left(\pi - \frac{\pi}{3}\right) \le \sin 2x \le \sin\left(\pi + \frac{\pi}{3}\right)$ or (A2) for correct ranges

$\sin\left(2\pi - \frac{\pi}{3}\right) \le \sin 2x \le \sin\left(2\pi + \frac{\pi}{3}\right)$

$-\frac{\pi}{3} \le 2x \le \frac{\pi}{3}$, $\frac{2\pi}{3} \le 2x \le \frac{4\pi}{3}$ or $\frac{5\pi}{3} \le 2x \le \frac{7\pi}{3}$ A1

$-\frac{\pi}{6} \le x \le \frac{\pi}{6}$, $\frac{\pi}{3} \le x \le \frac{2\pi}{3}$ or $\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$ (M1) for valid approach

$\therefore 0 \le x \le \frac{\pi}{6}$, $\frac{\pi}{3} \le x \le \frac{2\pi}{3}$ or $\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$ A3

[8]

6. (a) $E(X)=\int_{-2}^{3} x\cdot\frac{1}{5}\,dx$ (A1) for substitution

$E(X)=\left[\frac{1}{10}x^2\right]_{-2}^{3}$

$E(X)=\frac{9}{10}-\frac{4}{10}$

$E(X)=\frac{1}{2}$ A1

[2]

(b) $E(X^2)=\int_{-2}^{3} x^2\cdot\frac{1}{5}\,dx$ (A1) for substitution

$E(X^2)=\left[\frac{1}{15}x^3\right]_{-2}^{3}$

$E(X^2)=\frac{27}{15}-\left(-\frac{8}{15}\right)$

$E(X^2)=\frac{7}{3}$ A1

[2]

(c) Standard deviation

$=\sqrt{E(X^2)-(E(X))^2}$ (A1) for substitution

$=\sqrt{\frac{7}{3}-\left(\frac{1}{2}\right)^2}$

$=\sqrt{\frac{25}{12}}$ A1

[2]

7. (a) $f(|-x|)=\frac{7-2|-x|}{3-|-x|}$ M1

$f(|-x|)=\frac{7-2|x|}{3-|x|}$ A1

$f(|-x|)=f(|x|)$

Thus, $f(|x|)$ is an even function. AG

[2]

(b) $x=3$, $x=-3$ A2

[2]

(c) $y=-2$ A1

[1]

8. $2(\sec\alpha + 2\tan\alpha)^2 = 3 + 8\sec\alpha\tan\alpha + 6\tan^2\alpha$

$2(\sec^2\alpha + 4\sec\alpha\tan\alpha + 4\tan^2\alpha)$ (M1) for valid approach

$= 3 + 8\sec\alpha\tan\alpha + 6\tan^2\alpha$

$2\sec^2\alpha + 8\sec\alpha\tan\alpha + 8\tan^2\alpha$

$= 3 + 8\sec\alpha\tan\alpha + 6\tan^2\alpha$

$2\sec^2\alpha + 2\tan^2\alpha = 3$

$\sec^2\alpha + \tan^2\alpha = \frac{3}{2}$

$1 + \tan^2\alpha + \tan^2\alpha = \frac{3}{2}$ A1

$2\tan^2\alpha = \frac{1}{2}$

$\tan^2\alpha = \frac{1}{4}$

$\tan\alpha = -\frac{1}{2}$ or $\tan\alpha = \frac{1}{2}$ (*Rejected*) (A1) for correct value

$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$ (M1) for valid approach

$\tan 2\alpha = \frac{2\left(-\frac{1}{2}\right)}{1 - \left(-\frac{1}{2}\right)^2}$

$\tan 2\alpha = -\frac{4}{3}$ A1

[5]

9. When $n = 1$,

$1^3 + 3(1)^2 - 1 = 3$

$1^3 + 3(1)^2 - 1 = 3(1)$ A1

Thus, the statement is true when $n = 1$.

Assume that the statement is true when $n = k$. M1

$k^3 + 3k^2 - k = 3M$, where $M \in \mathbb{Z}$.

When $n = k + 1$,

$(k+1)^3 + 3(k+1)^2 - (k+1)$

$= k^3 + 3k^2 + 3k + 1 + 3(k^2 + 2k + 1) - k - 1$ M1

$= (3M + k - 3k^2) + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 - k - 1$ A1

$= 3M + 3k^2 + 9k + 3$ M1

$= 3(M + k^2 + 3k + 1)$, where $M + k^2 + 3k + 1 \in \mathbb{Z}$. A1

Thus, the statement is true when $n = k + 1$.

Therefore, the statement is true for all $n \in \mathbb{Z}^+$. R1

[7]

Section B

10. (a)

$g(x)-f(x)=0$

$e^{\frac{1}{2}\sqrt{x}}-e^{\frac{1}{2}\sqrt{x}}\sin\left(\frac{\pi}{3}x\right)=0$ (M1) for valid approach

$e^{\frac{1}{2}\sqrt{x}}\left(1-\sin\left(\frac{\pi}{3}x\right)\right)=0$

$1-\sin\left(\frac{\pi}{3}x\right)=0$

$\sin\left(\frac{\pi}{3}x\right)=1$ A1

$\frac{\pi}{3}x=\frac{\pi}{2}$, $\frac{\pi}{3}x=\frac{5\pi}{2}$ or $\frac{\pi}{3}x=\frac{9\pi}{2}$ (A1) for correct values

$x=\frac{3}{2}$, $x=\frac{15}{2}$ or $x=\frac{27}{2}$ A3

[6]

(b) (i)

$\frac{\pi}{3}x_n=\frac{\pi}{2}+(n-1)(2\pi)$ A1

$x_n=\frac{3}{2}+6(n-1)$

$x_{n+1}-x_n$

$=\left(\frac{3}{2}+6((n+1)-1)\right)-\left(\frac{3}{2}+6(n-1)\right)$ M1

$x_{n+1}-x_n=\left(\frac{3}{2}+6n\right)-\left(\frac{3}{2}+6n-6\right)$

$x_{n+1}-x_n=6$ A1

The differences between each pair of consecutive terms are equal to 6. Thus, $x_1, x_2, x_3, \ldots$ is an arithmetic sequence. AG

(ii)

$x_n=\frac{3}{2}+6n-6$

$x_n=6n-\frac{9}{2}$ A1

[4]

(c) Note that $x_2 = \frac{15}{2}$ and $x_3 = \frac{27}{2}$.

$f(x) = 0$

$e^{\frac{1}{2}\sqrt{x}} \sin\left(\frac{\pi}{3}x\right) = 0$ M1

$\sin\left(\frac{\pi}{3}x\right) = 0$

$\frac{\pi}{3}x = 3\pi$ or $\frac{\pi}{3}x = 4\pi$

$x = 9$ or $x = 12$ (A1) for correct values

$\therefore R = \int_{\frac{15}{2}}^{9}\left(e^{\frac{1}{2}\sqrt{x}} - e^{\frac{1}{2}\sqrt{x}}\sin\left(\frac{\pi}{3}x\right)\right)dx + \int_{9}^{12} e^{\frac{1}{2}\sqrt{x}}\,dx$

A2

$+\int_{12}^{\frac{27}{2}}\left(e^{\frac{1}{2}\sqrt{x}} - e^{\frac{1}{2}\sqrt{x}}\sin\left(\frac{\pi}{3}x\right)\right)dx$

[4]

11. (a) $\overrightarrow{PS}=\overrightarrow{PQ}+\overrightarrow{QS}$

$\overrightarrow{PS}=\mathbf{r}+\frac{1}{\alpha+1}\overrightarrow{QR}$ (A1) for correct approach

$\overrightarrow{PS}=\mathbf{r}+\frac{1}{\alpha+1}(\overrightarrow{PR}-\overrightarrow{PQ})$ (M1) for valid approach

$\overrightarrow{PS}=\mathbf{r}+\frac{1}{\alpha+1}(\mathbf{q}-\mathbf{r})$

$\overrightarrow{PS}=\frac{\alpha+1}{\alpha+1}\mathbf{r}+\frac{1}{\alpha+1}\mathbf{q}-\frac{1}{\alpha+1}\mathbf{r}$ (M1) for valid approach

$\overrightarrow{PS}=\frac{1}{\alpha+1}\mathbf{q}+\frac{\alpha}{\alpha+1}\mathbf{r}$ A1

[4]

(b) $\because \text{PS}\perp\text{QR}$

$\therefore \overrightarrow{PS}\cdot\overrightarrow{QR}=0$ M1

$\left(\frac{1}{\alpha+1}\mathbf{q}+\frac{\alpha}{\alpha+1}\mathbf{r}\right)\cdot(\mathbf{q}-\mathbf{r})=0$ A1

$(\mathbf{q}+\alpha\mathbf{r})\cdot(\mathbf{q}-\mathbf{r})=0$

$\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})+\alpha\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})=0$ M1

$\alpha\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})=-\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})$

$\alpha\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})=\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})$ M1

$\alpha=\frac{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}$ AG

[4]

(c) $\overrightarrow{PS}=\frac{1}{\alpha+1}\mathbf{q}+\frac{\alpha}{\alpha+1}\mathbf{r}$

$\overrightarrow{PS}=\frac{1}{\alpha+1}(\mathbf{q}+\alpha\mathbf{r})$

$\overrightarrow{PS}=\frac{1}{\frac{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}+1}\left(\mathbf{q}+\frac{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}\mathbf{r}\right)$ M1

$\overrightarrow{PS}=\frac{1}{\frac{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}+\frac{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}}\left(\frac{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}\mathbf{q}+\frac{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}\mathbf{r}\right)$ M1

$\overrightarrow{PS}=\frac{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})+\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}\left(\frac{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}\mathbf{q}+\frac{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}\mathbf{r}\right)$ A1

$$\vec{PS} = \frac{(\mathbf{r}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{r}}{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})+\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}$$ M1

$$\vec{PS} = \frac{(\mathbf{r}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{r}}{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})-\mathbf{r}\cdot(\mathbf{r}-\mathbf{q})}$$ A1

$$\vec{PS} = \frac{(\mathbf{r}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{r}}{(\mathbf{q}-\mathbf{r})\cdot(\mathbf{r}-\mathbf{q})}$$

$$\vec{PS} = -\frac{(\mathbf{r}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{r}}{|\mathbf{r}-\mathbf{q}|^2}$$ A1

$$\vec{PS} = \frac{(\mathbf{r}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{q}-(\mathbf{q}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{r}}{|\mathbf{r}-\mathbf{q}|^2}$$ AG

[6]

(d) (i)

$$\alpha = \frac{\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{r}\cdot(\mathbf{q}-\mathbf{r})}$$

$$\alpha = \frac{\mathbf{q}\cdot\mathbf{r}-\mathbf{q}\cdot\mathbf{q}}{\mathbf{r}\cdot\mathbf{q}-\mathbf{r}\cdot\mathbf{r}}$$ (M1) for valid approach

$$\alpha = \frac{\mathbf{q}\cdot\mathbf{r}-|\mathbf{q}|^2}{\mathbf{r}\cdot\mathbf{q}-|\mathbf{r}|^2}$$

$$\alpha = \frac{0-20^2}{0-15^2}$$ (A1) for substitution

$$\alpha = \frac{16}{9}$$ A1

(ii)

$$QR = \sqrt{20^2+15^2}$$

$$QR = 25$$ (A1) for correct value

$$RS = 25\left(\frac{\frac{16}{9}}{\frac{16}{9}+1}\right)$$

$$RS = 16$$ (A1) for correct value

$$PS = \sqrt{20^2-16^2}$$

$$PS = 12$$ (A1) for correct value

The required area

$$= \frac{(16)(12)}{2}$$

$$= 96$$ A1

[7]

12. (a)

$R^2 = \mathrm{OP}^2 + r^2$	(M1) for valid approach
$R^2 = (h-R)^2 + r^2$	
$R^2 = h^2 - 2Rh + R^2 + r^2$	
$2Rh - h^2 = r^2$	A1
$V = \frac{1}{3}\pi r^2 h$	
$\therefore V = \frac{1}{3}\pi(2Rh-h^2)(h)$	(A1) for substitution
$\therefore V = \frac{2R}{3}\pi h^2 - \frac{1}{3}\pi h^3$	A1

[4]

(b)

$V = \frac{2R}{3}\pi h^2 - \frac{1}{3}\pi h^3$	
$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{2R}{3}\pi(2h) - \frac{1}{3}\pi(3h^2)$	M1A1
$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{4R}{3}\pi h - \pi h^2$	A1
$\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = \frac{4R}{3}\pi(1) - \pi(2h)$	A1
$\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = \frac{4R}{3}\pi - 2\pi h$	AG

[4]

(c)

$\frac{\mathrm{d}V}{\mathrm{d}h} = 0$	
$\therefore \frac{4R}{3}\pi h - \pi h^2 = 0$	M1
$\frac{4R}{3} - h = 0$	
$h = \frac{4R}{3}$	A1
$\left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right\vert_{h=\frac{4R}{3}} = \frac{4R}{3}\pi - 2\pi\left(\frac{4R}{3}\right)$	M1
$\left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right\vert_{h=\frac{4R}{3}} = -\frac{4R}{3}\pi$	
$\left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right\vert_{h=\frac{4R}{3}} < 0$	R1

Thus, V attains its maximum when $h = \frac{4R}{3}$.

$2R\left(\frac{4R}{3}\right)-\left(\frac{4R}{3}\right)^2=r^2$ A1

$\frac{8R^2}{3}-\frac{16R^2}{9}=r^2$ M1

$\frac{8R^2}{9}=r^2$

$r=\frac{2\sqrt{2}R}{3}$

Thus, V attains its maximum when $r=\frac{2\sqrt{2}R}{3}$. AG

[6]

(d) $\frac{32}{81}\pi R^3$ A2

[2]

(e) The slant height of the circular cone

$=\sqrt{\left(\frac{2\sqrt{2}R}{3}\right)^2+\left(\frac{4R}{3}\right)^2}$ (M1) for valid approach

$=\sqrt{\frac{24}{9}R^2}$

$=\frac{\sqrt{24}R}{3}$

$=\frac{2\sqrt{6}R}{3}$ A1

The curved surface area of the circular cone

$=\pi\left(\frac{2\sqrt{2}R}{3}\right)\left(\frac{2\sqrt{6}R}{3}\right)$

$=\frac{4}{9}\sqrt{12}\pi R^2$

$<\frac{4}{9}(4)\pi R^2$ R1

$=\frac{16}{9}\pi R^2$

Thus, the curved surface area of the circular cone is not greater than $\frac{16}{9}\pi R^2$ when its volume attains its maximum. A1

[4]

AA HL Practice Set 3 Paper 2 Solution

Section A

1. (a) (i) 6 A1

(ii) 6 A1

(iii) The range

$=18-3$ (M1) for valid approach

$=15$ A1

[4]

(b) (i) The mean

$$=\frac{(3)(12)+(6)(20)+(9)(12)+(12)(8)+(15)(4)+(18)(4)}{12+20+12+8+4+4}$$ (M1) for valid approach

$=8.2$ A1

(ii) The variance

$=4.308131846^2$ (M1) for valid approach

$=18.6$ A1

[4]

2. (a) $f(x)=g(x)$

$\pi e^{-x^2}=1+\frac{1}{\pi e^{-x^2}}$ (M1) for setting equation

$\pi e^{-x^2}-1-\frac{e^{x^2}}{\pi}=0$

By considering the graph of $y=\pi e^{-x^2}-1-\frac{e^{x^2}}{\pi}$,

$x=-0.814566$ or $x=0.8145662$.

$\therefore a=-0.815$, $b=0.815$ A2

[3]

(b) The required area

$=\int_{-0.814566}^{0.8145662}(f(x)-g(x))\mathrm{d}x$ (A1) for correct integral

$=\int_{-0.814566}^{0.8145662}\left(\pi e^{-x^2}-1-\frac{e^{x^2}}{\pi}\right)\mathrm{d}x$

$=1.890606422$ (A1) for correct value

$=1.89$ A1

[3]

3. Note that $f(0)=-1$.

$-1=\sqrt{2}\sin\left(\frac{\pi}{6}(0+h)\right)$ (M1) for setting equation

$-\frac{1}{\sqrt{2}}=\sin\left(\frac{\pi}{6}h\right)$ (A1) for correct approach

$\frac{\pi}{6}h=-\frac{3\pi}{4}$ or $\frac{\pi}{6}h=-\frac{\pi}{4}$ (A1) for correct approach

$h=-4.5$ (*Rejected*) or $h=-1.5$ A1

$\therefore h=-1.5$ A1

[5]

4. (a) (i) $\frac{1}{2}$ A1

(ii) 3 A1

(iii) –4 A1

[3]

(b) The coordinates of P'

$= \left(\frac{2}{2}+3, -5(8-4)\right)$ (A2) for correct approach

$= (4, -20)$ A2

[4]

5. (a) $\cos\theta = \frac{AB}{r}$

$AB = r\cos\theta$ A1

[1]

(b) $\sin\theta = \frac{AE}{r}$

$AE = r\sin\theta$ A1

The area of the triangle ABE

$= \frac{(AB)(AE)}{2}$

$= \frac{(r\cos\theta)(r\sin\theta)}{2}$ M1

$= \frac{1}{2}r^2\sin\theta\cos\theta$ A1

$= \frac{1}{2}r^2\left(\frac{1}{2}\sin 2\theta\right)$ A1

$= \frac{r^2\sin 2\theta}{4}$ AG

[4]

(c) $A\hat{E}B + B\hat{E}C + C\hat{E}D = \pi$ M1

$\left(\frac{\pi}{2}-\theta\right) + B\hat{E}C + \left(\frac{\pi}{2}-\theta\right) = \pi$ A1

$\pi - 2\theta + B\hat{E}C = \pi$

$B\hat{E}C = 2\theta$ AG

[2]

6. (a) Let $\frac{x^2+2x+4}{(x-3)(x-7)} \equiv A+\frac{B}{x-3}+\frac{C}{x-7}$, where A, B and C are constants.

$\frac{x^2+2x+4}{(x-3)(x-7)} \equiv \frac{A(x-3)(x-7)}{(x-3)(x-7)}$ M1

$+\frac{B(x-7)}{(x-3)(x-7)}+\frac{C(x-3)}{(x-3)(x-7)}$

$\frac{x^2+2x+4}{(x-3)(x-7)}$

$\equiv \frac{Ax^2-10Ax+21A+Bx-7B+Cx-3C}{(x-3)(x-7)}$

x^2+2x+4

$\equiv Ax^2+(-10A+B+C)x+(21A-7B-3C)$ A1

$A=1$ A1

$2=-10(1)+B+C$

$C=12-B$

$4=21A-7B-3C$

$\therefore 4=21(1)-7B-3(12-B)$ A1

$4=21-7B-36+3B$

$19=-4B$

$B=-\frac{19}{4}$ A1

$\therefore C=12-\left(-\frac{19}{4}\right)$

$C=\frac{67}{4}$ A1

[6]

(b) $y=1$ A1

[1]

7. (a) (i) $\begin{cases} x+2y-z=1 \\ 2x-y+az=0 \\ x+3y+2z=b \end{cases}$

$\rightarrow \begin{cases} x+2y-z=1 \\ -5y+(a+2)z=-2 \\ y+3z=b-1 \end{cases}$ M1

$(R_2-2R_1 \,\&\, R_3-R_1)$

$\rightarrow \begin{cases} x+2y-z=1 \\ -5y+(a+2)z=-2 \quad (R_3+0.2R_2) \\ (0.2a+3.4)z=b-1.4 \end{cases}$ A1

The system has no solutions when

$0.2a+3.4=0$ and $b-1.4\neq 0$.

$a=-17$ and $b\neq 1.4$ A1

(ii) The system has a unique solution when

$0.2a+3.4\neq 0$.

$\therefore a\neq -17$ and $b\in\mathbb{R}$ A1

[4]

(b) $\begin{cases} x+2y-z=1 \\ 2x-y+3z=0 \\ x+3y+2z=3 \end{cases}$

By solving the system, $x=-0.2$, $y=0.8$ and

$z=0.4$. A2

[2]

8. $\mathbf{r}=(-1+2\lambda+4\mu)\mathbf{i}+(3+\lambda)\mathbf{j}+(-1+5\mu)\mathbf{k}$

$\mathbf{r}=(-\mathbf{i}+3\mathbf{j}-\mathbf{k})+\lambda(2\mathbf{i}+\mathbf{j})+\mu(4\mathbf{i}+5\mathbf{k})$ (M1) for valid approach

$\mathbf{n}=(2\mathbf{i}+\mathbf{j})\times(4\mathbf{i}+5\mathbf{k})$

$\mathbf{n}=\begin{pmatrix} (1)(5)-(0)(0) \\ (0)(4)-(2)(5) \\ (2)(0)-(1)(4) \end{pmatrix}$

$\mathbf{n}=5\mathbf{i}-10\mathbf{j}-4\mathbf{k}$ (A1) for correct values

The Cartesian equation of the plane π:

$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(5\mathbf{i}-10\mathbf{j}-4\mathbf{k})=(-\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(5\mathbf{i}-10\mathbf{j}-4\mathbf{k})$ M1A1

$5x-10y-4z=(-1)(5)+(3)(-10)+(-1)(-4)$

$5x-10y-4z=-31$ A1

[5]

9. (a) $f(0) = \arctan \frac{\pi}{2}(0) = 0$ (A1) for correct value

$$f'(x) = \left(\frac{1}{1 + \left(\frac{\pi}{2} x \right)^2} \right) \left(\frac{\pi}{2} \right)$$

$$f'(x) = \frac{2\pi}{4 + \pi^2 x^2}$$

$$f'(0) = \frac{2\pi}{4 + \pi^2 (0)^2} = \frac{\pi}{2}$$ (A1) for correct value

$$f''(x) = \frac{(4 + \pi^2 x^2)(0) - (2\pi)(2\pi^2 x)}{(4 + \pi^2 x^2)^2}$$ (M1) for valid approach

$$f''(x) = -\frac{4\pi^3 x}{(4 + \pi^2 x^2)^2}$$

$$f''(0) = -\frac{4\pi^3 (0)}{(4 + \pi^2 (0)^2)^2} = 0$$ (A1) for correct value

$$f^{(3)}(x) = -\frac{(4 + \pi^2 x^2)^2 (4\pi^3) - (4\pi^3 x)(2)(4 + \pi^2 x^2)(2\pi^2 x)}{(4 + \pi^2 x^2)^4}$$ (M1) for valid approach

$$f^{(3)}(0) = -\frac{(4+0)^2 (4\pi^3) - 0}{4^4} = -\frac{\pi^3}{4}$$ (A1) for correct value

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f^{(3)}(0) + \cdots$$

$$f(x) = 0 + x\left(\frac{\pi}{2} \right) + \frac{x^2}{2}(0) + \frac{x^3}{6}\left(-\frac{\pi^3}{4} \right) + \cdots$$

$$f(x) = \frac{\pi}{2} x - \frac{\pi^3}{24} x^3 + \cdots$$ A1

[7]

Section B

10. (a) $a = -0.0807147258$

$a = -0.0807$ A1

$b = 3.177202711$

$b = 3.18$ A1

[2]

(b) $\log y = -0.0807147258\sqrt{9} + 3.177202711$ (M1) for valid approach

$\log y = 2.935058534$

$y = 10^{2.935058534}$ (M1) for valid approach

$y = 861.1098035$

$y = 861$ A1

[3]

(c) $\log y = -0.0807147258\sqrt{x} + 3.177202711$

$y = 10^{-0.0807147258\sqrt{x}+3.177202711}$ (M1) for valid approach

$y = 10^{-0.0807147258\sqrt{x}} \cdot 10^{3.177202711}$ (A1) for correct approach

$y = 10^{3.177202711} \cdot (10^{-0.0807147258})^{\sqrt{x}}$ A1

$k = 10^{3.177202711}$ (A1) for correct approach

$k = 1503.843735$

$k = 1500$ A1

$m = 10^{-0.0807147258}$ (A1) for correct approach

$m = 0.8303960491$

$m = 0.830$ A1

[7]

11. (a) $a = \frac{v^2 + 64}{240}$

$\frac{dv}{dt} = \frac{v^2 + 64}{240}$

$\frac{1}{v^2 + 64} dv = \frac{1}{240} dt$ (M1) for valid approach

$\int \frac{1}{v^2 + 64} dv = \int \frac{1}{240} dt$ (A1) for correct approach

$\frac{1}{8} \arctan \frac{v}{8} = \frac{1}{240} t + C$ A1

$\arctan \frac{v}{8} = \frac{1}{30} t + C$

$\frac{v}{8} = \tan\left(\frac{1}{30} t + C\right)$

$v = 8 \tan\left(\frac{1}{30} t + C\right)$ A1

$0 = 8 \tan\left(\frac{1}{30}(0) + C\right)$ (M1) for substitution

$C = 0$ (A1) for correct value

$\therefore v = 8 \tan \frac{1}{30} t$ A1

[7]

(b) $\arctan \frac{v}{8} = \frac{1}{30} t$

$\arctan\left(\frac{1}{8} \cdot \frac{8}{3}\sqrt{3}\right) = \frac{1}{30} t$ (M1) for setting equation

$\arctan \frac{\sqrt{3}}{3} = \frac{1}{30} t$

$\frac{\pi}{6} = \frac{1}{30} t$ (A1) for correct approach

$t = 5\pi$ s A1

[3]

(c) $\frac{dv}{dt}=\frac{v^2+64}{240}$

$\frac{dv}{ds}\cdot\frac{ds}{dt}=\frac{v^2+64}{240}$ A1

$v\frac{dv}{ds}=\frac{v^2+64}{240}$ A1

$\frac{240v}{v^2+64}dv=ds$ M1

$\int\frac{240v}{v^2+64}dv=\int ds$ A1

$s=\int\frac{240v}{v^2+64}dv$ AG

[4]

(d) $s=\int\frac{240v}{v^2+64}dv$

Let $u=v^2+64$. (M1) for substitution

$\frac{du}{dv}=2v\Rightarrow 240v dv=120du$

$\therefore s=\int\frac{1}{u}\cdot 120du$ A1

$s=120\ln|u|+D$

$s=120\ln(v^2+64)+D$ A1

$0=120\ln(0^2+64)+D$ (M1) for substitution

$D=-120\ln 64$ (A1) for correct value

$\therefore s=120\ln(v^2+64)-120\ln 64$

$s=120\ln\left(\left(\frac{8}{3}\sqrt{3}\right)^2+64\right)-120\ln 64$

$s=34.52184869\text{ m}$

$s=34.5\text{ m}$ A1

[6]

12. (a) $\left(\cos\frac{\theta}{7}+i\sin\frac{\theta}{7}\right)^7$

$=\cos^7\frac{\theta}{7}+\binom{7}{1}i\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}+\binom{7}{2}i^2\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}$

$+\binom{7}{3}i^3\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}+\binom{7}{4}i^4\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}$

$+\binom{7}{5}i^5\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}+\binom{7}{6}i^6\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}$ A2

$+i^7\sin^7\frac{\theta}{7}$

$=\cos^7\frac{\theta}{7}+7i\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}-21\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}$

$-35i\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}+35\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}$ A1

$+21i\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}-7\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}-i\sin^7\frac{\theta}{7}$

$\therefore \cos\theta+i\sin\theta$

$=\cos^7\frac{\theta}{7}+7i\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}-21\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}$

$-35i\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}+35\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}$ M1

$+21i\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}-7\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}-i\sin^7\frac{\theta}{7}$

$=\cos^7\frac{\theta}{7}-21\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}$

$+35\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}-7\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}$

$+i\begin{pmatrix}7\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}-35\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}\\+21\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}-\sin^7\frac{\theta}{7}\end{pmatrix}$

$\therefore \cos\theta=\cos^7\frac{\theta}{7}-21\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}$
$+35\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}-7\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}$ and

$\sin\theta=7\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}-35\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}$
$+21\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}-\sin^7\frac{\theta}{7}$. A2

[6]

(b) $\tan\theta$

$$=\frac{\sin\theta}{\cos\theta}$$

$$=\frac{7\cos^6\frac{\theta}{7}\sin\frac{\theta}{7}-35\cos^4\frac{\theta}{7}\sin^3\frac{\theta}{7}+21\cos^2\frac{\theta}{7}\sin^5\frac{\theta}{7}-\sin^7\frac{\theta}{7}}{\cos^7\frac{\theta}{7}-21\cos^5\frac{\theta}{7}\sin^2\frac{\theta}{7}+35\cos^3\frac{\theta}{7}\sin^4\frac{\theta}{7}-7\cos\frac{\theta}{7}\sin^6\frac{\theta}{7}}$$ M1A1

$$=\frac{7\tan\frac{\theta}{7}-35\tan^3\frac{\theta}{7}+21\tan^5\frac{\theta}{7}-\tan^7\frac{\theta}{7}}{1-21\tan^2\frac{\theta}{7}+35\tan^4\frac{\theta}{7}-7\tan^6\frac{\theta}{7}}$$ A1

Let $x=\tan\frac{\theta}{7}$.

$$\tan\theta=\frac{7x-35x^3+21x^5-x^7}{1-21x^2+35x^4-7x^6}$$ M1

$$x^6-21x^4+35x^2-7=0$$

$$\frac{-x(x^6-21x^4+35x^2-7)}{1-21x^2+35x^4-7x^6}=0$$

$$\frac{7x-35x^3+21x^5-x^7}{1-21x^2+35x^4-7x^6}=0$$

$\tan\theta=0$ M1

$\theta=0$, π, 2π, 3π, 4π, 5π or 6π

$\therefore x=\tan\frac{0}{7}$, $x=\tan\frac{\pi}{7}$, $x=\tan\frac{2\pi}{7}$, $x=\tan\frac{3\pi}{7}$,

$x=\tan\frac{4\pi}{7}$, $x=\tan\frac{5\pi}{7}$ or $x=\tan\frac{6\pi}{7}$ A1

$x=0$ (*Rejected*), $x=\tan\frac{\pi}{7}$, $x=\tan\frac{2\pi}{7}$,

$x=\tan\frac{3\pi}{7}$, $x=\tan\frac{4\pi}{7}$, $x=\tan\frac{5\pi}{7}$ or $x=\tan\frac{6\pi}{7}$ A1

Thus, the equation $x^6-21x^4+35x^2-7=0$ has six roots. AG

[7]

(c) (i) $\sum_{r=1}^{7} \tan \frac{r\pi}{7}$

$= \sum_{r=1}^{6} \tan \frac{r\pi}{7} + \tan \frac{7\pi}{7}$ M1

$= -\frac{0}{1} + 0$ A1

$= 0$ A1

(ii) $\left(\tan \frac{\pi}{7}\right)\left(\tan \frac{2\pi}{7}\right)\left(\tan \frac{3\pi}{7}\right)$ M1A1

$\left(\tan \frac{4\pi}{7}\right)\left(\tan \frac{5\pi}{7}\right)\left(\tan \frac{6\pi}{7}\right) = -7$

$\left(\tan \frac{\pi}{7}\right)\left(\tan \frac{2\pi}{7}\right)\left(\tan \frac{3\pi}{7}\right)\left(\tan\left(\pi - \frac{3\pi}{7}\right)\right)$

$\left(\tan\left(\pi - \frac{2\pi}{7}\right)\right)\left(\tan\left(\pi - \frac{\pi}{7}\right)\right) = -7$

$\left(\tan \frac{\pi}{7}\right)\left(\tan \frac{2\pi}{7}\right)\left(\tan \frac{3\pi}{7}\right)\left(-\tan \frac{3\pi}{7}\right)$ A1

$\left(-\tan \frac{2\pi}{7}\right)\left(-\tan \frac{\pi}{7}\right) = -7$

$\left(\tan \frac{\pi}{7} \tan \frac{2\pi}{7} \tan \frac{3\pi}{7}\right)^2 = 7$

$\therefore \tan \frac{\pi}{7} \tan \frac{2\pi}{7} \tan \frac{3\pi}{7} = \sqrt{7}$ A1

[7]

AA HL Practice Set 3 Paper 3 Solution

1. (a) $r_1 + r_2 = -\frac{a_1}{1}$ (A1) for substitution

$a_1 = -r_1 - r_2$ A1

$r_1 r_2 = \frac{a_0}{1}$ (A1) for substitution

$a_0 = r_1 r_2$ A1

[4]

(b) (i) a_1

$= -r_1 - r_2$

$= -(r_1 + r_2)$

$= -S_1$ A1

(ii) $\frac{S_1^2 - S_2}{2}$

$= \frac{(r_1 + r_2)^2 - (r_1^2 + r_2^2)}{2}$

$= \frac{r_1^2 + 2r_1 r_2 + r_2^2 - r_1^2 - r_2^2}{2}$ M1A1

$= \frac{2r_1 r_2}{2}$

$= r_1 r_2$ A1

$= a_0$

$\therefore a_0 = \frac{S_1^2 - S_2}{2}$ AG

[4]

(c) (i) $a_2 = -S_1$ A1

(ii) $\frac{S_1^{\,2}-S_2}{2}$

$= \frac{(r_1+r_2+r_3)^2-(r_1^{\,2}+r_2^{\,2}+r_3^{\,2})}{2}$

$= \frac{r_1^{\,2}+r_1r_2+r_1r_3+r_1r_2+r_2^{\,2}+r_2r_3+r_1r_3+r_2r_3+r_3^{\,2}-r_1^{\,2}-r_2^{\,2}-r_3^{\,2}}{2}$ M1A1

$= \frac{2r_1r_2+2r_1r_3+2r_2r_3}{2}$

$= r_1r_2+r_1r_3+r_2r_3$ R1

$= a_1$

$\therefore a_1 = \frac{S_1^{\,2}-S_2}{2}$ is true. A1

[5]

(d) $ka_0 = S_1^{\,3}-3S_1S_2+2S_3$

$k(-r_1r_2r_3) = (r_1+r_2+r_3)^3$ (A1) for correct approach

$-3(r_1+r_2+r_3)(r_1^{\,2}+r_2^{\,2}+r_3^{\,2})+2(r_1^{\,3}+r_2^{\,3}+r_3^{\,3})$

$-kr_1r_2r_3 = (r_1+r_2+r_3)(r_1+r_2+r_3)^2$

$-3(r_1+r_2+r_3)(r_1^{\,2}+r_2^{\,2}+r_3^{\,2})+2(r_1^{\,3}+r_2^{\,3}+r_3^{\,3})$

$-kr_1r_2r_3$

$= (r_1+r_2+r_3)(r_1^{\,2}+r_2^{\,2}+r_3^{\,2}+2r_1r_2+2r_1r_3+2r_2r_3)$ M1A1

$-3(r_1+r_2+r_3)(r_1^{\,2}+r_2^{\,2}+r_3^{\,2})+2(r_1^{\,3}+r_2^{\,3}+r_3^{\,3})$

The coefficient of $r_1r_2r_3$ on R.H.S.

$=2+2+2$ A1

$=6$

$\therefore k=-6$ A1

[5]

(e) $a_{n-1} = -S_1$ A1

$a_{n-2} = \frac{S_1^{\,2}-S_2}{2}$ A1

$a_{n-3} = -\frac{1}{6}S_1^{\,3}+\frac{1}{2}S_1S_2-\frac{1}{3}S_3$ A1

[3]

(f) $\begin{cases} u+v+w=14 \\ u^2+v^2+w^2=86 \\ u^3+v^3+w^3=560 \end{cases}$

Let $S_1=14$, $S_2=86$ and $S_3=560$. (M1) for valid approach

u, v and w are the roots of the equation

$x^3+a_2x^2+a_1x+a_0=0$, where $a_2=-S_1$,

$a_1=\dfrac{S_1^{\,2}-S_2}{2}$ and $a_0=-\dfrac{1}{6}S_1^{\,3}+\dfrac{1}{2}S_1S_2-\dfrac{1}{3}S_3$. A1

$a_2=-14$ A1

$a_1=\dfrac{14^2-86}{2}$

$a_1=55$ A1

$a_0=-\dfrac{1}{6}(14)^3+\dfrac{1}{2}(14)(86)-\dfrac{1}{3}(560)$

$a_0=-42$ A1

Therefore, u, v and w are the roots of the

equation $x^3-14x^2+55x-42=0$. R1

By considering the graph of

$y=x^3-14x^2+55x-42$, $x=1$, $x=6$ or $x=7$.

$\therefore u=1$, $v=6$, $w=7$ A3

[9]

2. (a) $\cos(A+B)x + \cos(A-B)x$

$\cos Ax\cos Bx - \sin Ax\sin Bx$

$+\cos Ax\cos Bx + \sin Ax\sin Bx$ A2

$= 2\cos Ax\cos Bx$ AG

[2]

(b) $\int_0^\pi \cos Ax\cos Bx \mathrm{d}x$

$= \frac{1}{2}\int_0^\pi (\cos(A+B)x + \cos(A-B)x)\mathrm{d}x$ (A1) for substitution

$= \frac{1}{2}\left[\frac{1}{A+B}\sin(A+B)x + \frac{1}{A-B}\sin(A-B)x\right]_0^\pi$ A1

$= \frac{1}{2}\left[\begin{array}{l}\left(\frac{1}{A+B}\sin(A+B)\pi + \frac{1}{A-B}\sin(A-B)\pi\right) \\ -\left(\frac{1}{A+B}\sin 0 + \frac{1}{A-B}\sin 0\right)\end{array}\right]$ M1

$= 0$ A1

[4]

(c) (i) $\frac{1}{z}$

$= \frac{1}{\cos\theta + \mathrm{i}\sin\theta}$

$= \frac{\cos\theta - \mathrm{i}\sin\theta}{(\cos\theta + \mathrm{i}\sin\theta)(\cos\theta - \mathrm{i}\sin\theta)}$ M1

$= \frac{\cos\theta - \mathrm{i}\sin\theta}{\cos^2\theta - \mathrm{i}^2\sin^2\theta}$ A1

$= \frac{\cos\theta - \mathrm{i}\sin\theta}{\cos^2\theta + \sin^2\theta}$

$= \cos\theta - \mathrm{i}\sin\theta$ A1

(ii) $z + \frac{1}{z} = (\cos\theta + \mathrm{i}\sin\theta) + (\cos\theta - \mathrm{i}\sin\theta)$ (M1) for valid approach

$z + \frac{1}{z} = 2\cos\theta$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ A1

[5]

(d) $\cos^3\theta$

$$=\left(\frac{1}{2}\right)^3\left(z+\frac{1}{z}\right)^3$$

$$=\frac{1}{8}\left(z^3+\binom{3}{1}z^2\cdot\frac{1}{z}+\binom{3}{2}z\cdot\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3\right)$$ M1A1

$$=\frac{1}{8}\left(\begin{array}{l}\cos 3\theta+\mathrm{i}\sin 3\theta+3\cos\theta+3\mathrm{i}\sin\theta\\ +\dfrac{3}{\cos\theta+\mathrm{i}\sin\theta}+\dfrac{1}{\cos 3\theta+\mathrm{i}\sin 3\theta}\end{array}\right)$$ A1

$$=\frac{1}{8}\left(\begin{array}{l}\cos 3\theta+\mathrm{i}\sin 3\theta+3\cos\theta+3\mathrm{i}\sin\theta\\ +3(\cos\theta-\mathrm{i}\sin\theta)+\cos 3\theta-\mathrm{i}\sin 3\theta\end{array}\right)$$ A1

$$=\frac{1}{8}(2\cos 3\theta+6\cos\theta)$$

$$=\frac{1}{4}(\cos 3\theta+3\cos\theta)$$ AG

[4]

(e) $\cos^n\theta$

$$=\left(\frac{1}{2}\right)^n\left(z+\frac{1}{z}\right)^n$$

$$=\frac{1}{2^n}\left(\begin{array}{l}z^n+\dbinom{n}{1}z^{n-1}\cdot\dfrac{1}{z}+\dbinom{n}{2}z^{n-2}\cdot\left(\dfrac{1}{z}\right)^2\\ +\cdots+\dbinom{n}{n-1}z\cdot\left(\dfrac{1}{z}\right)^{n-1}+\left(\dfrac{1}{z}\right)^n\end{array}\right)$$ M1A1

$$=\frac{1}{2^n}\left(\begin{array}{l}z^n+\dbinom{n}{1}z^{n-2}+\dbinom{n}{2}z^{n-4}\\ +\cdots+\dbinom{n}{n-1}\dfrac{1}{z^{n-2}}+\dfrac{1}{z^n}\end{array}\right)$$ M1

$$=\frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\cos(n-2r)\theta$$ A2

[5]

(f) $\int_0^{\pi} \cos 6x \cos^5 x \mathrm{d}x$

$$= \int_0^{\pi} \cos 6x \left(\frac{1}{2^5} \sum_{r=0}^{5} \binom{5}{r} \cos(5-2r)x \right) \mathrm{d}x$$ (A1) for substitution

$$= \frac{1}{32} \int_0^{\pi} \cos 6x \begin{pmatrix} \cos 5x + 5\cos 3x + 10\cos x \\ +10\cos(-x) + 5\cos(-3x) + \cos(-5x) \end{pmatrix} \mathrm{d}x$$ M1

$$= \frac{1}{32} \int_0^{\pi} \cos 6x \begin{pmatrix} \cos 5x + 5\cos 3x + 10\cos x \\ +10\cos x + 5\cos 3x + \cos 5x \end{pmatrix} \mathrm{d}x$$ A1

$$= \frac{1}{32} \int_0^{\pi} \cos 6x (2\cos 5x + 10\cos 3x + 20\cos x) \mathrm{d}x$$

$$= \frac{1}{32} \int_0^{\pi} 2\cos 6x \cos 5x \mathrm{d}x + \frac{1}{32} \int_0^{\pi} 10\cos 6x \cos 3x \mathrm{d}x + \frac{1}{32} \int_0^{\pi} 20\cos 6x \cos x \mathrm{d}x$$

$$= \frac{1}{16} \int_0^{\pi} \cos 6x \cos 5x \mathrm{d}x + \frac{5}{16} \int_0^{\pi} \cos 6x \cos 3x \mathrm{d}x + \frac{5}{8} \int_0^{\pi} \cos 6x \cos x \mathrm{d}x$$ A1

$$= \frac{1}{16}(0) + \frac{5}{16}(0) + \frac{5}{8}(0)$$

$$= 0$$ A1

[5]

AA HL Practice Set 4 Paper 1 Solution

Section A

1. (a) The area of the shaded region

$=\frac{1}{2}(20)^2(1.5)$ (A1) for substitution

$=300\text{ cm}^2$ A1

[2]

(b) The arc length ABC

$=(20)(1.5)$ (A1) for substitution

$=30\text{ cm}$ A1

[2]

(c) The required perimeter

$=2\pi(20)-30+20+20$ (M1) for valid approach

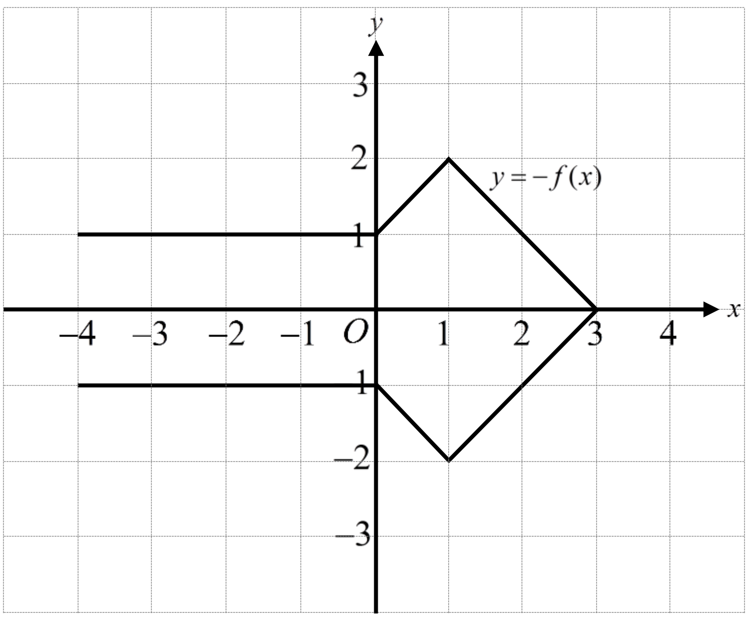
$=(40\pi+10)\text{ cm}$ A1

[2]

2. (a) For correct x-intercept and y-intercept A1

For two correct points $(-4, 1)$ and $(1, 2)$ A1

[2]



(b) $p=2$ A2

$q=-1$ A2

[4]

3. (a) $\log_4 64$

$= \log_4 4^3$ — (A1) for correct approach

$= 3$ — A1

[2]

(b) $\log_{12} 36 + \log_{12} 4$

$= \log_{12} 144$ — (A1) for correct approach

$= \log_{12} 12^2$

$= 2$ — A1

[2]

(c) $\log_2 11 - \log_2 88$

$= \log_2 \frac{1}{8}$ — (A1) for correct approach

$= \log_2 2^{-3}$

$= -3$ — A1

[2]

4. (a) $a = 2(-\sin \pi t)(\pi) + 0$ — (A1) for correct derivatives

$a = -2\pi \sin \pi t$ — A1

[2]

(b) $s = \int (2\cos \pi t + \pi) \mathrm{d}t$ — (M1) for indefinite integral

$s = \int 2\cos \pi t \mathrm{d}t + \int \pi \mathrm{d}t$

Let $u = \pi t$
$\frac{\mathrm{d}u}{\mathrm{d}t} = \pi \Rightarrow \frac{1}{\pi}\mathrm{d}u = \mathrm{d}t$

(A1) for substitution

$s = \int \frac{2}{\pi} \cos u \mathrm{d}u + \int \pi \mathrm{d}t$

$s = \frac{2}{\pi} \sin u + \pi t + C$ — A1

$s = \frac{2}{\pi} \sin \pi t + \pi t + C$

$\therefore -3 = \frac{2}{\pi} \sin 0 + 0 + C$ — (M1) for substitution

$C = -3$

$\therefore s = \frac{2}{\pi} \sin \pi t + \pi t - 3$ — A1

[5]

5. (a) $1 < D < 5$ A1

[1]

(b) 6 hours A1

[1]

(c) (i) The required mean

$= 10.5 + 1.5$ (M1) for valid approach

$= 12$ A1

(ii) The required variance

$= 2^2$ (M1)(A1) for correct approach

$= 4$ A1

[5]

6. $\lim_{x \to 0} \frac{1 + 3x - \cos\frac{\pi}{3}x}{\ln(1+x)}$

$= \lim_{x \to 0} \frac{0 + 3 - \left(-\sin\frac{\pi}{3}x\right)\left(\frac{\pi}{3}\right)}{\left(\frac{1}{x+1}\right)(1)} \left(\because \frac{0}{0}\right)$ M1A2

$= \lim_{x \to 0} (x+1)\left(3 + \frac{\pi}{3}\sin\frac{\pi}{3}x\right)$

$= (0+1)\left(3 + \frac{\pi}{3}\sin\frac{\pi}{3}(0)\right)$ M1

$= 3$ A1

[5]

7. $\tan x + \cot x + \dfrac{4\sqrt{3}}{3} = 0$

$\tan x + \cot x = -\dfrac{4\sqrt{3}}{3}$

$\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = -\dfrac{4\sqrt{3}}{3}$ (A1) for substitution

$\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x} = -\dfrac{4\sqrt{3}}{3}$

$1 = -\dfrac{4\sqrt{3}}{3} \sin x \cos x$ (M1) for valid approach

$-\sqrt{3} = 2(2 \sin x \cos x)$

$\sin 2x = -\dfrac{\sqrt{3}}{2}$ A1

$2x = \pi + \dfrac{\pi}{3}$ or $2x = 2\pi - \dfrac{\pi}{3}$

$x = \dfrac{2\pi}{3}$ or $x = \dfrac{5\pi}{6}$ A2

[5]

8. (a) $L_1 : \begin{cases} x = 17 + 5t \\ y = 1 - 2t \\ z = 10 + 3t \end{cases}$

$(17 + 5t) - 8 = 3 - (10 + 3t)$ (M1) for setting equation

$9 + 5t = -7 - 3t$

$16 = -8t$

$t = -2$ A1

$\therefore \begin{cases} x = 17 + 5(-2) = 7 \\ y = 1 - 2(-2) = 5 \\ z = 10 + 3(-2) = 4 \end{cases}$ (M1) for substitution

Thus, the coordinates of P are $(7, 5, 4)$. A1

[4]

(b) $\overrightarrow{\mathrm{RQ}} = -\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$

$\therefore \overrightarrow{\mathrm{OQ}} - \overrightarrow{\mathrm{OR}} = -\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$ (M1) for valid approach

$((17 + 5t)\mathbf{i} + (1 - 2t)\mathbf{j} + (10 + 3t)\mathbf{k}) - (3\mathbf{i} + 5\mathbf{k})$

$= -\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$

$17 + 5t - 3 = -1$

$5t = -15$

$t = -3$ (A1) for correct value

$\therefore \begin{cases} x = 17 + 5(-3) = 2 \\ y = 1 - 2(-3) = 7 \\ z = 10 + 3(-3) = 1 \end{cases}$ (M1) for substitution

Thus, the coordinates of Q are $(2, 7, 1)$. A1

[4]

9. When $n=1$,

$5-21(1)+4^1=-12$

$5-21(1)+4^1=3(-4)$ A1

Thus, the statement is true when $n=1$.

Assume that the statement is true when $n=k$. M1

$5-21k+4^k=3M$, where $M\in\mathbb{Z}$.

When $n=k+1$,

$5-21(k+1)+4^{k+1}$

$=5-21k-21+4(4^k)$ M1

$=-16-21k+4(3M+21k-5)$ A1

$=-16-21k+12M+84k-20$

$=12M+63k-36$ M1

$=3(4M+21k-12)$, where $4M+21k-12\in\mathbb{Z}$. A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

Section B

10. (a) (i) The required probability

$=\frac{3}{n}$ A1

(ii) The required probability

$=\left(\frac{n-3}{n}\right)\left(\frac{n-4}{n-1}\right)\left(\frac{3}{n-2}\right)$ (A1) for correct approach

$=\frac{3(n-3)(n-4)}{n(n-1)(n-2)}$ A1

[3]

(b) The required probability

$=\left(\frac{7}{10}\right)\left(\frac{6}{9}\right)\left(\frac{5}{8}\right)\left(\frac{3}{7}\right)$ (A1) for correct approach

$=\frac{1}{8}$ A1

[2]

(c) The game is fair if the expected gain is zero, which is equivalent to the expected amount of money earns back equals to $10. R1

$\therefore\left(\frac{3}{10}\right)(10)+\left(\left(\frac{7}{10}\right)\left(\frac{3}{9}\right)\right)(10)$

$+\left(\left(\frac{7}{10}\right)\left(\frac{6}{9}\right)\left(\frac{3}{8}\right)\right)(25x)+\left(\frac{1}{8}\right)(21x)$ M1A2

$+\left(1-\frac{3}{10}-\left(\frac{7}{10}\right)\left(\frac{3}{9}\right)-\left(\frac{7}{10}\right)\left(\frac{6}{9}\right)\left(\frac{3}{8}\right)-\frac{1}{8}\right)(0)=10$

$3+\frac{7}{3}+\frac{35}{8}x+\frac{21}{8}x=10$ M1A1

$7x=\frac{14}{3}$ A1

$x=\frac{2}{3}$ AG

[7]

11. (a) $z^{20}=1$

$z^{20}=\cos 0+\mathrm{i}\sin 0$ A1

$z=\cos\left(\frac{0+2k\pi}{20}\right)+\mathrm{i}\sin\left(\frac{0+2k\pi}{20}\right)$ M1

$(k=0,1,2,\cdots,18,19)$

$z=\cos 0+\mathrm{i}\sin 0$, $z=\cos\frac{\pi}{10}+\mathrm{i}\sin\frac{\pi}{10}$,

$z=\cos\frac{\pi}{5}+\mathrm{i}\sin\frac{\pi}{5}$, …,

$z=\cos\frac{9\pi}{5}+\mathrm{i}\sin\frac{9\pi}{5}$ or $z=\cos\frac{19\pi}{10}+\mathrm{i}\sin\frac{19\pi}{10}$ (A1) for correct values

$-\frac{\pi}{2}\le\arg(z)\le 0$

$\therefore z=\mathrm{cis}0$, $z=\mathrm{cis}\left(-\frac{\pi}{2}\right)$, $z=\mathrm{cis}\left(-\frac{2\pi}{5}\right)$,

$z=\mathrm{cis}\left(-\frac{3\pi}{10}\right)$, $z=\mathrm{cis}\left(-\frac{\pi}{5}\right)$ or $z=\mathrm{cis}\left(-\frac{\pi}{10}\right)$ A3

[6]

(b) $1+\cos\left(-\frac{2\pi}{5}\right)+\cos\left(-\frac{3\pi}{10}\right)$

$+\cos\left(-\frac{\pi}{5}\right)+\cos\left(-\frac{\pi}{10}\right)$ A1

[1]

(c) $\mathrm{Im}\,S$

$=-1+\sin\left(-\frac{2\pi}{5}\right)+\sin\left(-\frac{3\pi}{10}\right)$

$+\sin\left(-\frac{\pi}{5}\right)+\sin\left(-\frac{\pi}{10}\right)$ A1

$=-1-\sin\frac{2\pi}{5}-\sin\frac{3\pi}{10}-\sin\frac{\pi}{5}-\sin\frac{\pi}{10}$ M1

$=-1-\cos\left(\frac{\pi}{2}-\frac{2\pi}{5}\right)-\cos\left(\frac{\pi}{2}-\frac{3\pi}{10}\right)$

$-\cos\left(\frac{\pi}{2}-\frac{\pi}{5}\right)-\cos\left(\frac{\pi}{2}-\frac{\pi}{10}\right)$ A1

$=-1-\cos\frac{\pi}{10}-\cos\frac{\pi}{5}-\cos\frac{3\pi}{10}-\cos\frac{4\pi}{10}$ M1

$$= -\begin{pmatrix} 1 + \cos\left(-\frac{2\pi}{5}\right) + \cos\left(-\frac{3\pi}{10}\right) \\ + \cos\left(-\frac{\pi}{5}\right) + \cos\left(-\frac{\pi}{10}\right) \end{pmatrix}$$ A1

$$= -\operatorname{Re} S$$

$$\therefore \frac{\operatorname{Re} S}{\operatorname{Im} S} = -1$$ AG

[5]

(d) (i) $\cos\left(-\frac{\pi}{5}\right)$

$$= \cos\frac{\pi}{5}$$

$$= 2\cos^2\frac{\pi}{10} - 1$$ (A1) for substitution

$$= 2\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 - 1$$

$$= \frac{10+2\sqrt{5}}{8} - 1$$ (M1) for valid approach

$$= \frac{10+2\sqrt{5}-8}{8}$$

$$= \frac{1+\sqrt{5}}{4}$$ A1

(ii) $\cos\left(-\frac{2\pi}{5}\right)$

$$= 2\cos^2\left(-\frac{\pi}{5}\right) - 1$$ (A1) for substitution

$$= 2\left(\frac{1+\sqrt{5}}{4}\right)^2 - 1$$

$$= \frac{1+2\sqrt{5}+5}{8} - 1$$ (M1) for valid approach

$$= \frac{6+2\sqrt{5}-8}{8}$$

$$= \frac{\sqrt{5}-1}{4}$$ A1

[6]

(e) $\operatorname{Im} S$

$= -\operatorname{Re} S$

$= -\begin{pmatrix} 1+\cos\left(-\frac{2\pi}{5}\right)+\cos\left(-\frac{3\pi}{10}\right) \\ +\cos\left(-\frac{\pi}{5}\right)+\cos\left(-\frac{\pi}{10}\right) \end{pmatrix}$ M1

$= -\left(1+\frac{\sqrt{5}-1}{4}+\frac{\sqrt{10-2\sqrt{5}}}{4}+\frac{1+\sqrt{5}}{4}+\frac{\sqrt{10+2\sqrt{5}}}{4}\right)$ A1

$= -\left(1+\frac{\sqrt{5}}{2}+\frac{\sqrt{10-2\sqrt{5}}+\sqrt{10+2\sqrt{5}}}{4}\right)$

$= -\left(\frac{4}{4}+\frac{2\sqrt{5}}{4}+\frac{\sqrt{10-2\sqrt{5}}+\sqrt{10+2\sqrt{5}}}{4}\right)$

$= -\frac{4+2\sqrt{5}+\sqrt{10-2\sqrt{5}}+\sqrt{10+2\sqrt{5}}}{4}$ AG

[2]

12. (a) (i) $f(x)=g(x)$

$\therefore \sin 2\pi y=-\sin \pi y$ M1

$2\sin \pi y\cos \pi y+\sin \pi y=0$ A1

$\sin \pi y(2\cos \pi y+1)=0$ A1

$\sin \pi y=0$ or $\cos \pi y=-\frac{1}{2}$

$\pi y=0$ or $\pi y=\frac{2\pi}{3}$ A1

$y=0$ (*Rejected*) or $y=\frac{2}{3}$

$\therefore r=\frac{2}{3}$ AG

(ii) The area of the region

$=\int_{\frac{2}{3}}^{1}(g(y)-f(y))\mathrm{d}y$ A1

$=\int_{\frac{2}{3}}^{1}(-\sin \pi y-\sin 2\pi y)\mathrm{d}y$

$=\left[\frac{1}{\pi}\cos \pi y+\frac{1}{2\pi}\cos 2\pi y\right]_{\frac{2}{3}}^{1}$ A1

$=\left(\frac{1}{\pi}\cos \pi(1)+\frac{1}{2\pi}\cos 2\pi(1)\right)$
$-\left(\frac{1}{\pi}\cos \pi\left(\frac{2}{3}\right)+\frac{1}{2\pi}\cos 2\pi\left(\frac{2}{3}\right)\right)$ M1

$=\left(-\frac{1}{\pi}+\frac{1}{2\pi}\right)-\left(\frac{1}{\pi}\left(-\frac{1}{2}\right)+\frac{1}{2\pi}\left(-\frac{1}{2}\right)\right)$ A1

$=-\frac{1}{2\pi}-\left(-\frac{1}{2\pi}-\frac{1}{4\pi}\right)$ M1

$=-\frac{1}{2\pi}+\frac{1}{2\pi}+\frac{1}{4\pi}$

$=\frac{1}{4\pi}$ AG

[9]

(b)	$a\sin 2\pi\left(\frac{3}{4}\right)=-\frac{\sqrt{2}}{2}$	(M1) for substitution
	$-a=-\frac{\sqrt{2}}{2}$	A1
	$a=\frac{\sqrt{2}}{2}$	A1

[3]

(c)	$f(x)=g(x)$	
	$\therefore a\sin 2\pi y=-\sin\pi y$	M1
	$2a\sin\pi y\cos\pi y+\sin\pi y=0$	A1
	$\sin\pi y(2a\cos\pi y+1)=0$	A1
	$2a\cos\pi y+1=0$	M1
	$2a\cos\pi y=-1$	
	$\cos\pi y=-\frac{1}{2a}$	A1
	$\therefore \sin\pi y$	
	$=\sqrt{1-\cos^2\pi y}$	A1
	$=\sqrt{1-\left(-\frac{1}{2a}\right)^2}$	
	$=\sqrt{1-\frac{1}{4a^2}}$	M1
	$=\sqrt{\frac{4a^2-1}{4a^2}}$	A1
	$\pi y=\arcsin\sqrt{\frac{4a^2-1}{4a^2}}$	M1
	$\therefore r=\frac{1}{\pi}\arcsin\sqrt{\frac{4a^2-1}{4a^2}}$	AG

[9]

AA HL Practice Set 4 Paper 2 Solution

Section A

1. (a) (i) $(3, -127)$ A2

(ii) $f(x) = 3(x-3)^2 - 127$ A2

[4]

(b) $3x^2 - 18x - 100 = -52$

$3x^2 - 18x - 48 = 0$ (A1) for correct equation

$3(x+2)(x-8) = 0$

$x = -2$ or $x = 8$ A2

[3]

2. (a) p is negative as the first turning point is a minimum point. R1

$p = -\frac{4.3}{2}$ A1

$p = -2.15$ AG

[2]

(c) (i) The period

$= 13.75 - 2.75$ (M1) for valid approach

$= 11$ hours (A1) for correct value

$\therefore q = \frac{2\pi}{11}$ A1

(ii) $r = \frac{(1.9 + 4.3) + 1.9}{2}$ (M1) for valid approach

$r = 4.05$ A1

[5]

3. (a) $\mathrm{B\hat{A}C}$

$= \pi - 0.88 - 1.23$	(M1) for valid approach
$= 1.031592654$	A1
$\dfrac{\mathrm{AB}}{\sin \mathrm{A\hat{C}B}} = \dfrac{\mathrm{BC}}{\sin \mathrm{B\hat{A}C}}$	(M1) for sine rule
$\dfrac{\mathrm{AB}}{\sin 1.23} = \dfrac{20}{\sin 1.031592654}$	(A1) for substitution
$\mathrm{AB} = 21.96641928 \text{ cm}$	
$\mathrm{AB} = 22.0 \text{ cm}$	A1

[5]

(b)

$\mathrm{AB}^2 = \mathrm{OA}^2 + \mathrm{OB}^2 - 2(\mathrm{OA})(\mathrm{OB}) \cos \mathrm{A\hat{O}B}$	M1
$\mathrm{AB}^2 = r^2 + r^2 - 2(r)(r) \cos \mathrm{A\hat{O}B}$	A1
$\mathrm{AB}^2 = 2r^2 - 2r^2 \cos \mathrm{A\hat{O}B}$	
$\mathrm{AB}^2 = 2r^2 (1 - \cos \mathrm{A\hat{O}B})$	A1
$r^2 = \dfrac{\mathrm{AB}^2}{2(1 - \cos \mathrm{A\hat{O}B})}$	AG

[3]

4. (a) The common ratio r

$= \dfrac{3k^2 - 4k^3}{k^2}$	(M1) for valid approach
$= 3 - 4k$	A1

[2]

(b)

S_∞ exists if $-1 < r < 1$.	R1
$\therefore -1 < 3 - 4k < 1$	M1
$-1 < 4k - 3 < 1$	
$2 < 4k < 4$	A1
$\dfrac{1}{2} < k < 1$	AG

[3]

5. The general term

$= \binom{9}{r}\left(\frac{x}{h^2}\right)^{9-r}\left(-\frac{h}{x^2}\right)^r$ (M1) for valid expansion

$= \binom{9}{r}(-1)^r h^{3r-18} x^{9-3r}$

$9-3r=0$ (A1) for correct equation

$3r=9$

$r=3$ (A1) for correct value

The required term

$= \binom{9}{3}(-1)^3 h^{3(3)-18} x^{9-3(3)}$

$= -\frac{84}{h^9}$ (A1) for correct term

$-\frac{84}{h^9} = -\frac{21}{65536}$ (M1) for setting equation

$h^9 = 262144$

$h = 4$ A1

[6]

6. (a) Let $\frac{x^2+9}{(4-x)(5-2x)} \equiv A+\frac{B}{4-x}+\frac{C}{5-2x}$, where A, B and C are constants.

$\frac{x^2+9}{(4-x)(5-2x)} \equiv \frac{A(4-x)(5-2x)}{(4-x)(5-2x)}$ M1

$+\frac{B(5-2x)}{(4-x)(5-2x)}+\frac{C(4-x)}{(4-x)(5-2x)}$

$\frac{x^2+9}{(4-x)(5-2x)}$

$\equiv \frac{20A-13Ax+2Ax^2+5B-2Bx+4C-Cx}{(4-x)(5-2x)}$

$x^2+9 \equiv 2Ax^2+(-13A-2B-C)x+(20A+5B+4C)$ A1

$2A=1$

$A=\frac{1}{2}$ A1

$0=-13\left(\frac{1}{2}\right)-2B-C$

$C=-\frac{13}{2}-2B$

$9=20A+5B+4C$

$\therefore 9=20\left(\frac{1}{2}\right)+5B+4\left(-\frac{13}{2}-2B\right)$ A1

$9=10+5B-26-8B$

$25=-3B$

$B=-\frac{25}{3}$ A1

$\therefore C=-\frac{13}{2}-2\left(-\frac{25}{3}\right)$

$C=\frac{61}{6}$ A1

[6]

(b) $g(x)=-\frac{(4-x)(5-2x)}{x^2+9}$

The discriminant of x^2+9

$=0^2-4(1)(9)$ A1

$=-36$

<0

Therefore, the denominator is always nonzero.

Thus, $g(x)$ has no vertical asymptote. AG

[1]

7. (a) $\left\{x : -5 \leq x \leq \frac{2}{3}\right\}$ A2

[2]

(b) $f(x) = (3x-2)^2$

$y = (3x-2)^2$

$\Rightarrow x = (3y-2)^2$ (M1) for swapping variables

$-\sqrt{x} = 3y - 2$

$-\sqrt{x} + 2 = 3y$

$y = \dfrac{-\sqrt{x}+2}{3}$

$\therefore f^{-1}(x) = \dfrac{-\sqrt{x}+2}{3}$ A1

[2]

(c) $(f \circ g)(x) = x^4$

$g(x) = f^{-1}(x^4)$ M1

$g(x) = \dfrac{-\sqrt{x^4}+2}{3}$

$g(x) = \dfrac{-x^2+2}{3}$ A1

[2]

8. $\binom{12}{2} \times \binom{10}{r} \times \binom{10-r}{10-r} = 7920$ M1A1

$\binom{10}{r} = 120$ (A1) for simplification

$\binom{10}{r} = \binom{10}{3}$ or $\binom{10}{r} = \binom{10}{7}$

$r = 3$ or $r = 7$ A2

[5]

9. (a) The standard deviation of X

$=\sqrt{E(X^2)-(E(X))^2}$ (M1) for valid approach

$=\sqrt{\int_{-4}^{0} x^2\cdot\left(\frac{1}{20}x+\frac{1}{5}\right)dx+\int_{0}^{3} x^2\cdot\left(-\frac{1}{15}x^2+\frac{2}{15}x+\frac{1}{5}\right)dx-\left(\frac{13}{60}\right)^2}$ A1

$=\sqrt{2.279722222}$

$=1.509874903$

$=1.51$ A1

[3]

(b) $P(|X|>2)$

$=P(X>2 \text{ or } X<-2)$ (M1) for valid approach

$=P(X<-2)+P(X>2)$

$=\int_{-4}^{-2}\left(\frac{1}{20}x+\frac{1}{5}\right)dx+\int_{2}^{3}\left(-\frac{1}{15}x^2+\frac{2}{15}x+\frac{1}{5}\right)dx$ A1

$=\frac{19}{90}$ A1

[3]

Section B

10. (a) (i) $a_1(t) = \frac{20-30}{2-0}$ M1A1

$a_1(t) = -5$ AG

(ii) $v_1(t) = -5t + 30$ A2

[4]

(b) The total distance the marble travelled

$= \int_0^2 |v_1(t)| \, dt$ (M1) for valid approach

$= \int_0^2 |-5t+30| \, dt$ (A1) for correct formula

$= 50$ cm A1

[3]

(c) (i) $v_2(2) = 20$

$\therefore 20e^{b-0.2(2)} = 20$ M1

$e^{b-0.4} = 1$

$b - 0.4 = 0$ A1

$b = 0.4$ AG

(ii) $\int_2^c |v_2(t)| \, dt = 50$

$\int_2^c 20e^{0.4-0.2t} dt = 50$ (M1) for setting equation

Let $u = 0.4 - 0.2t$
$\frac{du}{dt} = -0.2 \Rightarrow -100du = 20dt$
$t = c \Rightarrow u = 0.4 - 0.2c$
$t = 2 \Rightarrow u = 0.4 - 0.2(2) = 0$

(A1) for substitution

$\int_0^{0.4-0.2c} -100e^u du = 50$ A1

$\left[-100e^u\right]_0^{0.4-0.2c} = 50$

$e^{0.4-0.2c} - e^0 = -0.5$ (M1) for substitution

$e^{0.4-0.2c} = 0.5$

$0.4 - 0.2c = \ln 0.5$

$0.4 - \ln 0.5 = 0.2c$

$c = 5.465735903$

$c = 5.47$ A1

[7]

11. (a) The coordinates of A, B' and C are $(-3, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 8)$ respectively. A1

$\mathbf{n} = \overrightarrow{\text{AB}'} \times \overrightarrow{\text{AC}}$ M1

$\mathbf{n} = (3\mathbf{i} + 4\mathbf{j}) \times (3\mathbf{i} + 8\mathbf{k})$ A1

$$\mathbf{n} = \begin{pmatrix} (4)(8) - (0)(0) \\ (0)(3) - (3)(8) \\ (3)(0) - (4)(3) \end{pmatrix}$$

$\mathbf{n} = 32\mathbf{i} - 24\mathbf{j} - 12\mathbf{k}$ A1

The Cartesian equation of the plane π_2:

$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (32\mathbf{i} - 24\mathbf{j} - 12\mathbf{k})$

$= -3\mathbf{i} \cdot (32\mathbf{i} - 24\mathbf{j} - 12\mathbf{k})$ M1A1

$32x - 24y - 12z = (-3)(32) + (0)(-24) + (0)(-12)$

$32x - 24y - 12z = -96$

$8x - 6y - 3z = -24$ AG

[6]

(b) The coordinates of B are $(0, -4, 0)$. (A1) for correct values

The volume of the pyramid ABCC'

$= \frac{1}{3}\left(\frac{(\text{BB}')(\text{OA})}{2}\right)(\text{OC})$ (M1) for valid approach

$= \frac{1}{3}\left(\frac{(4-(-4))(0-(-3))}{2}\right)(8)$ A1

$= 32$ A1

[4]

(c) $\mathbf{n}_1 = 8\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$

$\mathbf{n}_2 = 8\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ (A1) for correct values

Let θ be the obtuse angle between the planes.

$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1||\mathbf{n}_2|\cos\theta$ (M1) for valid approach

$(8)(8) + (6)(-6) + (-3)(-3)$

$= (\sqrt{8^2 + 6^2 + (-3)^2})(\sqrt{8^2 + (-6)^2 + (-3)^2})\cos\theta$ (A1) for substitution

$37 = (\sqrt{109})(\sqrt{109})\cos\theta$

$\cos\theta = \frac{37}{109}$ A1

$\theta = 70.15665929^\circ$

The required obtuse angle

$= 180^\circ - 70.15665929^\circ$

$= 109.8433407^\circ$

$= 110^\circ$ A1

[5]

(d) The mid-point of BC

$= \left(\frac{0+0}{2}, \frac{-4+0}{2}, \frac{0+8}{2}\right)$

$= (0, -2, 4)$ (A1) for correct values

$\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2$ (M1) for valid approach

$\mathbf{n}_3 = (8\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) \times (8\mathbf{i} - 6\mathbf{j} - 3\mathbf{k})$

$$\mathbf{n}_3 = \begin{pmatrix} (6)(-3) - (-3)(-6) \\ (-3)(8) - (8)(-3) \\ (8)(-6) - (6)(8) \end{pmatrix}$$

$\mathbf{n}_3 = -36\mathbf{i} - 96\mathbf{k}$ A1

The vector equation of the line:

$$\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} -36 \\ 0 \\ -96 \end{pmatrix}$$ A1

$$\begin{cases} x = -36t \\ y = -2 \\ z = 4 - 96t \end{cases}$$

$\frac{x}{-36} = \frac{z-4}{-96}, y = -2$ A1

[5]

12. (a) (i) $x^2\dfrac{dy}{dx}+6y=x^3e^{x^2+\frac{6}{x}}$

$\dfrac{dy}{dx}+\dfrac{6}{x^2}y=xe^{x^2+\frac{6}{x}}$ (A1) for correct approach

The integrating factor

$=e^{\int\frac{6}{x^2}dx}$ (M1) for valid approach

$=e^{-\frac{6}{x}}$ A1

$\therefore e^{-\frac{6}{x}}\dfrac{dy}{dx}+e^{-\frac{6}{x}}\cdot\dfrac{6}{x^2}y=e^{-\frac{6}{x}}\cdot xe^{x^2+\frac{6}{x}}$ (M1) for valid approach

$e^{-\frac{6}{x}}\dfrac{dy}{dx}+e^{-\frac{6}{x}}\cdot\dfrac{6}{x^2}y=xe^{x^2}$

$\dfrac{d}{dx}\left(ye^{-\frac{6}{x}}\right)=xe^{x^2}$ (A1) for correct approach

$ye^{-\frac{6}{x}}=\int xe^{x^2}dx$

Let $u=x^2$. (M1) for substitution

$\dfrac{du}{dx}=2x\Rightarrow\dfrac{1}{2}du=xdx$

$\therefore ye^{-\frac{6}{x}}=\int e^u\cdot\dfrac{1}{2}du$ (A1) for correct working

$ye^{-\frac{6}{x}}=\dfrac{1}{2}\int e^u du$

$ye^{-\frac{6}{x}}=\dfrac{1}{2}e^u+C$

$ye^{-\frac{6}{x}}=\dfrac{1}{2}e^{x^2}+C$ A1

$ye^{-\frac{6}{x}}=\dfrac{e^{x^2}+C}{2}$

$y=\dfrac{e^{\frac{6}{x}}(e^{x^2}+C)}{2}$ A1

$\dfrac{e^7}{2}=\dfrac{e^{\frac{6}{1}}(e^{1^2}+C)}{2}$ (M1) for substitution

$\dfrac{e^7}{2}=\dfrac{e^7+Ce^6}{2}$

$Ce^6=0$

$C=0$

$$\therefore y = \frac{e^{\frac{6}{x}+x^2}}{2}$$ A1

(ii) $\frac{e^{11}}{2}$ A1

[12]

(b) (i) $\begin{cases} x_{n+1} = x_n + 0.1 \\ y_{n+1} = y_n + 0.1\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\, y_n)} \end{cases}$ M1

$x_0 = 1,\ y_0 = \frac{e^7}{2}$ A1

$x_1 = 1 + 0.1$

$x_1 = 1.1$

$y_1 = \frac{e^7}{2} + 0.1\left(1e^{1^2+\frac{6}{1}} - \frac{6}{1^2}\left(\frac{e^7}{2}\right)\right)$ M1A1

$y_1 = \frac{e^7}{2} - 0.2e^7$

$y_1 = \frac{3e^7}{10}$ AG

(ii) 23435.5461 A2

[6]

(c) $23435.5461 < \frac{e^{11}}{2}$ R1

[1]

AA HL Practice Set 4 Paper 3 Solution

1. (a) $F(2)$

$= \sum_{r=1}^{2} \sin\frac{\pi}{2(2)}\sin\frac{r\pi}{2}$ (M1) for substitution

$= \sin\frac{\pi}{4}\sum_{r=1}^{2}\sin\frac{r\pi}{2}$

$= \sin\frac{\pi}{4}\left(\sin\frac{\pi}{2}+\sin\pi\right)$ A1

$= \left(\sin\frac{\pi}{4}\right)(1+0)$

$= \sin\frac{\pi}{4}$ A1

[3]

(b) (i) $\cos(x-y)-\cos(x+y)$

$= \cos x\cos y+\sin x\sin y$

$-(\cos x\cos y-\sin x\sin y)$ A2

$= 2\sin x\sin y$ AG

(ii) Let $x=\frac{A+B}{2}$ and $y=\frac{B-A}{2}$.

$\cos(x-y)-\cos(x+y)$

$= \cos\left(\frac{A+B}{2}-\frac{B-A}{2}\right)$

$-\cos\left(\frac{A+B}{2}+\frac{B-A}{2}\right)$ A1

$= \cos\frac{2A}{2}-\cos\frac{2B}{2}$ M1

$= \cos A-\cos B$

$\therefore \cos A-\cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$ AG

[4]

(c) $F(4)$

$$=\sum_{r=1}^{4}\sin\frac{\pi}{2(4)}\sin\frac{r\pi}{4}$$

$$=\sin\frac{\pi}{8}\sum_{r=1}^{4}\sin\frac{r\pi}{4}$$

$$=\sin\frac{\pi}{8}\left(\sin\frac{\pi}{4}+\sin\frac{\pi}{2}+\sin\frac{3\pi}{4}+\sin\pi\right)$$ A1

$$=\sin\frac{\pi}{8}\sin\frac{\pi}{4}+\sin\frac{\pi}{8}\sin\frac{\pi}{2}$$
$$+\sin\frac{\pi}{8}\sin\frac{3\pi}{4}+\sin\frac{\pi}{8}\sin\pi$$

$$=\frac{1}{2}\left(\cos\left(\frac{\pi}{8}-\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{8}+\frac{\pi}{4}\right)\right)$$
$$+\frac{1}{2}\left(\cos\left(\frac{\pi}{8}-\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{8}+\frac{\pi}{2}\right)\right)$$ M1A1
$$+\frac{1}{2}\left(\cos\left(\frac{\pi}{8}-\frac{3\pi}{4}\right)-\cos\left(\frac{\pi}{8}+\frac{3\pi}{4}\right)\right)$$

$$=\frac{1}{2}\left(\begin{array}{l}\cos\left(-\frac{\pi}{8}\right)-\cos\frac{3\pi}{8}+\cos\left(-\frac{3\pi}{8}\right)-\cos\frac{5\pi}{8}\\+\cos\left(-\frac{5\pi}{8}\right)-\cos\frac{7\pi}{8}\end{array}\right)$$

$$=\frac{1}{2}\left(\begin{array}{l}\cos\frac{\pi}{8}-\cos\frac{3\pi}{8}+\cos\frac{3\pi}{8}-\cos\frac{5\pi}{8}\\+\cos\frac{5\pi}{8}-\cos\frac{7\pi}{8}\end{array}\right)$$ A1

$$=\frac{1}{2}\left(\cos\frac{\pi}{8}-\cos\frac{7\pi}{8}\right)$$

$$=\frac{1}{2}\left(2\sin\frac{\frac{\pi}{8}+\frac{7\pi}{8}}{2}\sin\frac{\frac{7\pi}{8}-\frac{\pi}{8}}{2}\right)$$ A1

$$=\sin\frac{\pi}{2}\sin\frac{3\pi}{8}$$

$$=\sin\frac{3\pi}{8}$$ A1

[6]

(d) $F(n)$

$$= \sum_{r=1}^{n} \sin\frac{\pi}{2n}\sin\frac{r\pi}{n}$$

$$= \sum_{r=1}^{n} \frac{1}{2}\left(\cos\left(\frac{\pi}{2n}-\frac{r\pi}{n}\right)-\cos\left(\frac{\pi}{2n}+\frac{r\pi}{n}\right)\right)$$ M1A1

$$= \sum_{r=1}^{n} \frac{1}{2}\left(\cos\left(\frac{\pi}{2n}-\frac{2r\pi}{2n}\right)-\cos\left(\frac{\pi}{2n}+\frac{2r\pi}{2n}\right)\right)$$

$$= \sum_{r=1}^{n} \frac{1}{2}\left(\cos\frac{(1-2r)\pi}{2n}-\cos\frac{(1+2r)\pi}{2n}\right)$$ M1

$$= \frac{1}{2}\left(\begin{array}{l}\cos\frac{(1-2(1))\pi}{2n}-\cos\frac{(1+2(1))\pi}{2n} \\ +\cos\frac{(1-2(2))\pi}{2n}-\cos\frac{(1+2(2))\pi}{2n} \\ +\cdots+\cos\frac{(1-2n)\pi}{2n}-\cos\frac{(1+2n)\pi}{2n}\end{array}\right)$$

$$= \frac{1}{2}\left(\begin{array}{l}\cos\left(-\frac{\pi}{2n}\right)-\cos\frac{3\pi}{2n}+\cos\left(-\frac{3\pi}{2n}\right)-\cos\frac{5\pi}{2n} \\ +\cdots+\cos\frac{(1-2n)\pi}{2n}-\cos\frac{(1+2n)\pi}{2n}\end{array}\right)$$

$$= \frac{1}{2}\left(\begin{array}{l}\cos\frac{\pi}{2n}-\cos\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}-\cos\frac{5\pi}{2n} \\ +\cdots+\cos\frac{(2n-1)\pi}{2n}-\cos\frac{(1+2n)\pi}{2n}\end{array}\right)$$ A1

$$= \frac{1}{2}\left(\cos\frac{\pi}{2n}-\cos\frac{(1+2n)\pi}{2n}\right)$$ M1

$$= \frac{1}{2}\left(2\sin\frac{\frac{\pi}{2n}+\frac{(1+2n)\pi}{2n}}{2}\sin\frac{\frac{(1+2n)\pi}{2n}-\frac{\pi}{2n}}{2}\right)$$ A1

$$= \sin\frac{(2+2n)\pi}{4n}\sin\frac{2n\pi}{4n}$$

$$= \sin\frac{(1+n)\pi}{2n}\sin\frac{\pi}{2}$$

$$\therefore F(n) = \sin\frac{(1+n)\pi}{2n}$$ AG

[6]

(e) $|z^r - 1|$

$= \left|\left(\cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n}\right)^r - 1\right|$

$= \left|\cos\frac{2\pi r}{n} + \mathrm{i}\sin\frac{2\pi r}{n} - 1\right|$ (M1) for valid approach

$= \sqrt{\left(\cos\frac{2\pi r}{n} - 1\right)^2 + \sin^2\frac{2\pi r}{n}}$

$= \sqrt{\cos^2\frac{2\pi r}{n} - 2\cos\frac{2\pi r}{n} + 1 + \sin^2\frac{2\pi r}{n}}$ M1

$= \sqrt{2 - 2\cos\frac{2\pi r}{n}}$

$= \sqrt{2 - 2\left(1 - 2\sin^2\frac{\pi r}{n}\right)}$ A1

$= \sqrt{4\sin^2\frac{\pi r}{n}}$

$= 2\sin\frac{\pi r}{n}$

$\because -1 \le \sin\frac{\pi r}{n} \le 1$ R1

$\therefore |z^r - 1| \le 2$ A1

[5]

(f) $G(n)$

$$= \sum_{r=1}^{n} \left| z^r - 1 \right|$$

$$= \sum_{r=1}^{n} 2\sin\frac{\pi r}{n}$$

$$= \frac{2\sum_{r=1}^{n} \sin\frac{\pi}{2n}\sin\frac{\pi r}{n}}{\sin\frac{\pi}{2n}}$$ M1

$$= \frac{2F(n)}{\sin\frac{\pi}{2n}}$$ A1

$$= \frac{2\sin\frac{(1+n)\pi}{2n}}{\sin\frac{\pi}{2n}}$$

$$= \frac{2\cos\left(\frac{\pi}{2} - \frac{(1+n)\pi}{2n}\right)}{\sin\frac{\pi}{2n}}$$ A1

$$= \frac{2\cos\left(\frac{n\pi}{2n} - \frac{\pi + n\pi}{2n}\right)}{\sin\frac{\pi}{2n}}$$

$$= \frac{2\cos\left(-\frac{\pi}{2n}\right)}{\sin\frac{\pi}{2n}}$$ M1

$$= \frac{2\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}$$

$$= 2\cot\frac{\pi}{2n}$$ A1

[5]

2. (a) (i) $I(0)$

$= \int_0^{\pi} x\,\mathrm{d}x$ M1

$= \left[\frac{1}{2}x^2\right]_0^{\pi}$ A1

$= \frac{1}{2}\pi^2 - \frac{1}{2}(0)^2$

$= \frac{1}{2}\pi^2$ AG

(ii) $I(1)$

$= \int_0^{\pi} x\sin x\,\mathrm{d}x$

Let $\theta = \cos x$. (M1) for valid approach

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = -\sin x \Rightarrow (-1)\frac{\mathrm{d}\theta}{\mathrm{d}x} = \sin x$

$\therefore I(1)$

$= \int_0^{\pi} x(-1)\frac{\mathrm{d}(\cos x)}{\mathrm{d}x}\mathrm{d}x$

$= \left[-x\cos x\right]_0^{\pi} - \int_0^{\pi} \cos x \cdot \frac{\mathrm{d}(-x)}{\mathrm{d}x}\mathrm{d}x$ A1

$= \left[-x\cos x\right]_0^{\pi} - \int_0^{\pi} \cos x \cdot (-1)\mathrm{d}x$ A1

$= \left[-x\cos x\right]_0^{\pi} + \int_0^{\pi} \cos x\,\mathrm{d}x$

$= \left[-x\cos x\right]_0^{\pi} + \left[\sin x\right]_0^{\pi}$ A1

$= \left[-x\cos x + \sin x\right]_0^{\pi}$

$= (-\pi\cos\pi + \sin\pi) - (0 + \sin 0)$

$= \pi$ A1

[7]

(b) (i) $I(n+2)$

$= \int_0^{\pi} x\sin^{n+2} x\,\mathrm{d}x$

$= \int_0^{\pi} x\sin^{n} x\sin^2 x\,\mathrm{d}x$ M1

$= \int_0^{\pi} x\sin^{n} x(1-\cos^2 x)\mathrm{d}x$

$= \int_0^{\pi} x\sin^{n} x\,\mathrm{d}x - \int_0^{\pi} x\sin^{n} x\cos^2 x\,\mathrm{d}x$ A1

$= I(n) - \int_0^{\pi} x\sin^{n} x\cos^2 x\,\mathrm{d}x$ AG

(ii) $\int_0^{\pi} x\sin^n x\cos^2 x\mathrm{d}x$

$$=\frac{1}{n+1}\int_0^{\pi} x\cos x\cdot\frac{\mathrm{d}(\sin^{n+1}x)}{\mathrm{d}x}\mathrm{d}x$$

$$=\frac{1}{n+1}\left\{\begin{aligned}&\left[x\cos x\sin^{n+1}x\right]_0^{\pi}\\&-\int_0^{\pi}\sin^{n+1}x\cdot\frac{\mathrm{d}(x\cos x)}{\mathrm{d}x}\mathrm{d}x\end{aligned}\right\}$$ A1

$$=\frac{1}{n+1}\left\{\begin{aligned}&\left[x\cos x\sin^{n+1}x\right]_0^{\pi}\\&-\int_0^{\pi}\sin^{n+1}x(\cos x-x\sin x)\mathrm{d}x\end{aligned}\right\}$$ A1

$$=\frac{1}{n+1}\left\{\begin{aligned}&\left[x\cos x\sin^{n+1}x\right]_0^{\pi}\\&-\int_0^{\pi}(\sin^{n+1}x\cos x-x\sin^{n+2}x)\mathrm{d}x\end{aligned}\right\}$$

$$=\frac{1}{n+1}\left\{\begin{aligned}&\left[x\cos x\sin^{n+1}x\right]_0^{\pi}\\&-\int_0^{\pi}\sin^{n+1}x\cos x\mathrm{d}x+I(n+2)\end{aligned}\right\}$$ M1

$$=\frac{1}{n+1}\left\{\begin{aligned}&(\pi\cos\pi\sin^{n+1}\pi-0)\\&-\int_0^{\pi}\sin^{n+1}x\cos x\mathrm{d}x+I(n+2)\end{aligned}\right\}$$ M1

$$=\frac{1}{n+1}\left\{-\int_0^{\pi}\sin^{n+1}x\cos x\mathrm{d}x+I(n+2)\right\}$$

Let $u=\sin x$. (M1) for substitution

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\cos x\Rightarrow\mathrm{d}u=\cos x\mathrm{d}x$$

$x=\pi\Rightarrow u=\sin\pi=0$

$x=0\Rightarrow u=\sin 0=0$

$\therefore\int_0^{\pi} x\sin^n x\cos^2 x\mathrm{d}x$

$$=\frac{1}{n+1}\left\{-\int_0^{0}u^{n+1}\mathrm{d}u+I(n+2)\right\}$$ A1

$$=\frac{1}{n+1}(0+I(n+2))$$

$$=\frac{1}{n+1}I(n+2)$$ A1

(iii) $I(n+2) = I(n) - \frac{1}{n+1} I(n+2)$ A1

$(n+1)I(n+2) = (n+1)I(n) - I(n+2)$

$(n+2)I(n+2) = (n+1)I(n)$ M1

$I(n+2) = \frac{n+1}{n+2} I(n)$ AG

[11]

(c) (i) $I(4)$

$= \frac{2+1}{2+2} I(2)$ M1

$= \frac{3}{4}\left(\frac{0+1}{0+2} I(0)\right)$ M1

$= \frac{3}{4}\left(\frac{1}{2} \cdot \frac{1}{2} \pi^2\right)$

$= \frac{3}{16} \pi^2$ A1

(ii) $I(7)$

$= \frac{5+1}{5+2} I(5)$ M1

$= \frac{6}{7}\left(\frac{3+1}{3+2} I(3)\right)$ M1

$= \frac{6}{7}\left(\frac{4}{5}\right)\left(\frac{1+1}{1+2} I(1)\right)$

$= \frac{6}{7}\left(\frac{4}{5}\right)\left(\frac{2}{3} \pi\right)$

$= \frac{16}{35} \pi$ A1

[6]

(d) $0 \le \sin x \le 1$ for $0 \le x \le \pi$. A1

Therefore, $\sin^2 x \le \sin x \le 1$, implies that

$\int_0^\pi x \sin^{2n-2} x \cdot \sin^2 x dx$

$\le \int_0^\pi x \sin^{2n-2} x \cdot \sin x dx \le \int_0^\pi x \sin^{2n-2} x \cdot 1 dx$. R1

Thus, $I(2n) \le I(2n-1) \le I(2n-2)$ for $n \ge 1$. AG

[2]